Some basis:

PROOF BY CONTRADICTION: proof opposite statement false, therefore original statement true.

Sets:

SET: collection of ELEMENTS: objects in a set.

$A\&B$ sets:

Name	Notation	Meaning
	$x \in A$	x an element of A
	$x \notin A$	x not an elemnt of A
UNION	$A \cup B$	$x \in A$ and/or $x \in B$
INTERSECTION	$A \cap B$	$x \in A$ and $\in B$
EMPTY SET:	$\emptyset$	Set contains no element.
E and S are DISJOINT	$E \cap S = \emptyset$	
COMPLEMENT OF A	$A^c = \{x \in \mathbb{R} : x \notin A\}$	the set of all elements in R, but not in A
SUBSET	$A \subseteq B$	All elements in A are also elements in B
SUPSET	$B \supseteq A$	B contains all the elements of A
	$A = B$	When $A \subseteq B$ and $B \subseteq A$
De Morgan's Law	$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$	Proof? **Exercise 1.2.5**

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ all elements of A_2 also elements of A_1 and so on (so A_{n+1} elements of A_n)

Functions and real numbers:

$A\&B$ are sets, a, b real numbers.
Definition 1.2.3: Functions:
FUNCTION: from A to B maps each element $x \in A$ with a single element of B
`Notation:` $f : A \to B$ given $x \in A$ and expression $f(x)$ represents element B assiociate with x by f
DOMAIN: A&RANGE: subset of B given by: $\{y \in B : y = f(x)$ for some $x \in A\}$

Theorem 1.2.6:
a, b equal iff for every real number $\varepsilon > 0$, it follows $|a - b| < \varepsilon$
PROOF:
(1):If $a = b$ then $|a - b| < \varepsilon$
$|a - b| = 0$ and because $\varepsilon > 0$ we know $|a - b| < \varepsilon$
(2): If $|a - b| < \varepsilon$ then $a = b$
Assume $a \neq b$ so $\varepsilon_0 = |a - b| > 0$ must be true, which is the case because $\varepsilon > 0$
But $|a - b| < \varepsilon_0$ and $|a - b| = \varepsilon_0$ can not be both true.
Therefore $a \neq b$ unacceptable $\Rightarrow a = b$

INDUCTION:
If $S \subset \mathbb{N}$ with: $1 \in S$ $\quad n \in \mathbb{N}$ and $n \in S$ $\quad n + 1 \in S$ then $S = \mathbb{N}$

Lecture 1:

Lemma and proof:

$|x| = \max\{x, -x\}$

Definition of an absolute value: $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

proof:

$x > 0 \Rightarrow -x \leq 0 \Rightarrow -x \leq x \Rightarrow \max\{-x, x\} = x = |x|$

$x < 0 \Rightarrow -x > 0 \Rightarrow -x > x \Rightarrow \max -x, x = -x = |x|$

Algebraic properties:

Name	Rule	Proof:			
		x	y	xy	conclusion
Product rule	$\lvert xy\rvert = \lvert x\rvert \cdot \lvert y\rvert$	$x > 0$	$y > 0$	$xy > 0$	$\lvert xy\rvert = xy = \lvert x\rvert \cdot \lvert y\rvert$
		$x > 0$	$y < 0$	$xy < 0$	$\lvert xy\rvert = x(-y) = \lvert x\rvert \cdot \lvert y\rvert$
		$x < 0$	$y > 0$	$xy < 0$	$\lvert xy\rvert = (-x)y = \lvert x\rvert \cdot \lvert y\rvert$
		$x < 0$	$y < 0$	$xy > 0$	$\lvert xy\rvert = (-x)(-y) = \lvert x\rvert \cdot \lvert y\rvert$
Quotient rule	$\left\lvert \frac{x}{y} \right\rvert = \frac{\lvert x\rvert}{\lvert y\rvert}$ where $y \neq 0$	Proof by yourself. it is sufficient to show that $\left\lvert \frac{1}{y} \right\rvert = \frac{1}{\lvert y\rvert}$			
	$\lvert a-b\rvert = \lvert b-a\rvert$	$\lvert a-b\rvert = \lvert -(b-a)\rvert = \lvert b-a\rvert$			

Inequalities:

Name	Rule	Proof
Lemma 2	$\lvert x\rvert \leq a \Leftrightarrow -a \leq x \leq a$	$\lvert x\rvert \leq a \Leftrightarrow \max\{-x, x\} \leq a$ $\Leftrightarrow -x \leq a \text{ and } x \leq a$ $\Leftrightarrow x \geq -a \text{ and } x \leq a$ $\Leftrightarrow -a \leq x \leq a$
Triangle inequality	$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	$x + y \leq \lvert x\rvert + y \leq \lvert x\rvert + \lvert y\rvert$ $-x - y \leq \lvert x\rvert - y \leq \lvert x\rvert + \lvert y\rvert$ $\lvert x+y\rvert = \max\{x+y, -x-y\} \leq \lvert x\rvert + \lvert y\rvert$
Reverse triangle Inequality	$\lvert\lvert x\rvert - \lvert y\rvert\rvert \leq \lvert x-y\rvert$	$\lvert x\rvert = \lvert x-y+y\rvert \leq \lvert x-y\rvert + \lvert y\rvert$ $\lvert x\rvert - \lvert y\rvert \leq \lvert x-y\rvert$ $\lvert y\rvert - \lvert x\rvert \leq \lvert y-x\rvert = \lvert x-y\rvert$ $\lvert\lvert x\rvert - \lvert y\rvert\rvert = \max\{\lvert x\rvert - \lvert y\rvert, \lvert y\rvert - \lvert x\rvert\}$ $\leq \lvert x-y\rvert$

Upper bounds:

Name	Bounded above	Least upper bound
Definition	$A \subseteq \mathbb{R}$ is bounded above if: $\exists b \in \mathbb{R}$ s.t. $a \leq b$ and $\forall a \in A$	$s \in \mathbb{R}$ least upper bound of $A \subseteq \mathbb{R}$ if: s upper bound A b any upper bound A, and $s \leq b$
Notation	the number b is called an upper bound	$s = \sup(A)$ called the supremum of the set A
Example	$A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \}$ $b \geq 1$ upper bound for A	$A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \}$ Claim: $\sup(A) = 1$ Clearly, $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$ so 1 is an upper bound for A if b is any upper bound for A then $a \leq b$ for all $a \in A$ in particular, for $a = 1$ we have $1 \leq b$
Number	**Definition 1.3.1**	**Definition 1.3.2**

Lemma 1.3.8:
if s is an upper bound for A then: $s = \sup A \leftrightarrow \forall \varepsilon > 0 \exists a \in A$ s.t. $s - \varepsilon < a$
Proof part 1:

Proof part 1:	Proof part 2:
Let $\varepsilon > 0$ arbitrary	Let b upper bound for A
$s - \varepsilon < s \to s = \varepsilon$ not upper bound A	$b < s$ then for $\varepsilon = s - b$ exists $a \in A$ s.t. $b = s - \varepsilon < a$
$\exists a \in A$ s.t $s = \varepsilon < a$	b not upper bound, contradiction.
	Hence $s \leq b$ implies $s = \sup(A)$

Lower bounds

Name	Lower bound:	Greatest lower bound
Definition	l is called a lower bound of $A \subseteq \mathbb{R}$ if: $\exists l \in \mathbb{R}$ s.t. $l \leq a \forall a \in A$	$i \in \mathbb{R}$ is called the greatest lower bound of $A \subseteq \mathbb{R}$ if: i lower bound for A and l any lower bound for A where $l \leq i$
Notation		$i = \inf(A)$
Example	$\{\frac{1}{n} : n \in \mathbb{N}\}$ any number $l \leq 0$ lower bound for A	
Number	**Definition 1.3.1**	

Lemma 4:
if i is a lower bound for A then: $i = \inf A \leftrightarrow \forall \varepsilon > 0 \exists a \in A$ s.t. $a < i + \varepsilon$
Proof: **Exercise 1.3.1**

Maximum and minimum:

Definition 1.3.4 Maximum and minimum: real number a_0 maximum of set A if a_0 element of A and $a_0 \geq a$ for all $a \in A$
real number a_1 minimum of A if $a_1 \leq a$ for all $a \in A$

Warning: $\sup(A)$ not always maximum A .For example $\sup\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \} = 1$ no largest element!
$\inf(A)$ not always minimum A. For example $\inf\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} = 0$, no smallest element.

Lecture 2

The real line:

Name	Set	Ordening($<,=,>$)?	Algebraic operations?
Natural numbers	$\mathbb{N}$	Yes	$+\times$
Integers	$\mathbb{Z}$	Yes	$+-\times$
Rational numbers	$\mathbb{Q}$	Yes	$+-\times:$
Real numbers	$\mathbb{R}$	Yes	$+-\times:$

What is the difference between $\mathbb{Q}$ and $\mathbb{R}$?
$\mathbb{Q}$ has many gaps. Numbers like $\sqrt{2}, e, \pi$ are not in $\mathbb{Q}$

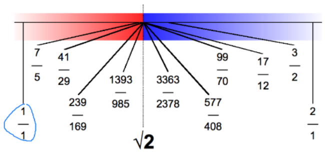
Example:

By example that $\sqrt{2} \notin \mathbb{Q}$

Theorem	$\sqrt{2} \notin \mathbb{Q}$
Proof:	Assume $\sqrt{2} = \frac{p}{q}$, with $p, q \in \mathbb{Z}$ and $\text{GCD}(p,q) = 1$
	$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$
	So p^2 is even, so p is even, say $p = 2k$
	$\Rightarrow p^2 = 2q^2 \Rightarrow (2k)^2 = 2q^2 \rightarrow q^2 = 2k^2$
	q^2 is even so q is even.
	$\text{GCD}(p,q) \neq 1$, at least 2
	so proven by contradiction $\sqrt{2} \neq \frac{p}{q}$ so $\sqrt{2} \notin \mathbb{Q}$

Do least upper bounds exist?

We used the definitions we saw in the first lecture for least upper bound and greatest lower bound.



Red: the set $A = \{x \in \mathbb{Q} : x \leq 2\}$
Blue: the upper bounds for A that are in $\mathbb{Q}$
Is this subset bounded above? Therefore we use a new axiom.

Definitions:

Axiom of Completeness (AoC): Every nonempty set of $\mathbb{R}$ is bounded above has a least upper bound.
Theorem 1.4.2: Archimendean property:
Consist 2 parts:

Theorem	$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n > x$	$\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$
Proof	Not true? $\mathbb{N}$ bounded above	Let $y > 0$ arbitrary
	AOC $\Rightarrow \alpha = \sup \mathbb{N}$ where $\alpha \notin \mathbb{N}$	Set $x = \frac{1}{y}$
	$\alpha - 1$ not upper bound.	By the first statement, exists $N \in \mathbb{N}$ s.t. $n > x$
	Exists $n \in \mathbb{N}$ s.t. $\alpha - 1 < n \Rightarrow \alpha < n + 1$	Therefore $\frac{1}{n} < \frac{1}{x} = y$
	$n + 1 \in \mathbb{N} \Rightarrow \alpha$ Not upper bound $\mathbb{N}$	
	Contradiction.	

Nested Interval Property closed interval:

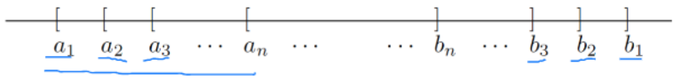
Theorem 1.4.1:

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots \to \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$

PROOF:

We have to show that $\exists x \in \mathbb{R}$ s.t $x \in [a_n, b_n] \forall n \in \mathbb{N}$

Define $A = \{a_n : n \in \mathbb{N}\}$



so we see that b_n upper bound a_n

AoC gives us: $x := \sup(A)$ exists.

$a_n \leq x \quad \forall n \in \mathbb{N} \quad$ Since x =upper bound for A

$x \leq b_n \quad \forall n \in \mathbb{N} \quad$ Since $x =$ least upper bound of A

$x \in [a_n, b_n] \quad \forall n \in \mathbb{N}$

Nested Interval Property open interval:

The NIP does not work for open intervals:

EXAMPLE:

Proof that for $I_n = (0, \frac{1}{n})$ we have that $\bigcap_{n=1}^{\infty} I_n = \emptyset$

When $x \leq 0$ we have $x \notin I_n$ forall $n \in \mathbb{N}$

When $x > 0$ we have that $\exists k \in \mathbb{N}$ s.t. $\frac{1}{k} < x$(by AP), And therefore,$\exists k \in \mathbb{N}$ s.t. $x \notin I_k$

So in both cases we have $x \notin \bigcap_{n=1}^{\infty} I_n$ so $\bigcap_{n=1}^{\infty} I_n = \emptyset$

Rational and Real numbers:

Theorem 1.4.3: $\forall a, b \in \mathbb{R}$ with $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$

PROOF:

(1) $a < 0 < b$ then one nice r between it, namely the rational number 0

(2) $0 \leq a < b$ (works also for $b < a \leq 0$, by working with $-a$ and $-b$)

$\exists, n, m \in \mathbb{N}$ s.t.

$$\left.\begin{matrix} \frac{1}{n} < b - a \\ m - 1 \leq na < m \end{matrix}\right\} \Rightarrow m \leq na + 1 < n(b - \tfrac{1}{n}) + 1 = nb$$

Combine inequalities.

$$\left.\begin{matrix} na < m \\ m < mb \end{matrix}\right\} \Rightarrow na < m < nb \Rightarrow a < \tfrac{m}{n} < b$$

$\frac{m}{n} \in \mathbb{Q}$ so there exists indeed $r \in \mathbb{Q}$ s.t. $a < r < b$

Existence of square roots:

$\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$

PROOF:

define $A = \{t \in \mathbb{R} : t^2 \leq 2\}$ and $\alpha = \sup A$,then:

$\alpha^2 < 2$ take $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1}$	$\alpha^2 > 2$ take $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{\alpha^2}{2\alpha}$
So $(\alpha + \frac{1}{n})^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \leq \alpha^2 + \frac{2\alpha+1}{n} < 2$	$(\alpha - \frac{1}{n})^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} > 2$
So $\alpha + \frac{1}{n} \in A$ so α not upper bound A	Also contradiction, therefore, the theorem is true

Lecture 3

1-1 CORRESPONDENCE: counting without counting by making sets.

Functions:

Definition:

FUNCTION: $f : A \to B$ maps each $a \in A$ with single element $b = f(a) \in B$.
DOMAIN: A&RANGE: $\text{ran}(f) = f(A) = \{f(a) : a \in A\}$&CODOMAIN: B
Types:
INJECTIVE (ONE-TO ONE) if $f(a) = f(b) \to a = b$
SURJECTIVE (ONTO) if $B = f(A)$ i.e. $\forall b \in B \exists a \in A$ s.t. $b = f(a)$
BIJECTIVE: if f injective and surjective (unique corerespondence between elements of $A\&B$)

Allowed and not allowed.
Two elements in domain can correspond to 1 element in the codomain.
All elements in the domain must correspond to some element in the codomain.
An element in the domain can not correspond to more then 1 element in the codomain.

Cardinality:
Two sets same cardinality if there exists a bijective function: $f : A \to B$
Notation: $A \sim B$
So 1 to one correspondence, so equally many elements in both sets.
If $\sim$ equivalence relation:
$A \sim A$
$A \sim B \leftrightarrow B \sim A$
$A \sim B$ and $B \sim C \Rightarrow A \sim C$
PROOF:
$(a, b) \sim (1, 1)$ condsider.
$g : (a, b) \to (-1, 1)$ so $g(x) = \frac{2x-a-b}{b-a}$
Use $(a, b) \sim \mathbb{R}$ and $(-1, 1) \sim \mathbb{R}$ so $(a, b) \sim (-1, 1)$

Example:

1:
$\mathbb{N} = \{1, 2, 3, \ldots\} \sim \mathbb{E} = \{2, 4, 6, \ldots\}$
A bijection is given by: $f : \mathbb{N} \to \mathbb{E}$ so:
$f(n) = 2n$
Moral: there are "as many" even numbers as natural numbers.

2:
$\mathbb{N} \sim \mathbb{Z}$
A bijection (exercise) is given by:
$f : \mathbb{N} \to \mathbb{Z}$
$f(n) = \begin{cases} (n-1)/2 \text{ if } n \text{ is odd} \\ -n/2 \text{ if } n \text{ is even} \end{cases}$
Moral: there are "as many" integers as natural numbers!

3:
to prove that $(-1,1) \sim \mathbb{R}$ consider:
$f : (-1,1) \to \mathbb{R}$ and $f(x) = \frac{x}{1-x^2}$

f is injective:
$f(a) = f(b) \leftrightarrow a(1-b^2) = b(1-a)^2 \leftrightarrow a - b + a^2b - ab^2 = 0 \leftrightarrow (a-b)(ab+1) = 0$
$(ab+1)$ can not be zero (because of the domain) $\to a - b = 0 \to a = b$
Note: $a, b \in (-1,1) \to ab \in (-1,1)$

f is surjective:
$f(x) = r \leftrightarrow x = r(1-x^2) \leftrightarrow rx^2 + x - r = 0$ is solvible for all $r \in \mathbb{R}$
Note: discriminant $= 1 + 4r^2 > 0$
$x = \frac{-1 \pm \sqrt{1+4r^2}}{2r}$
These equation has 2 solutions.
For any $r \in \mathbb{R}$ has unique solution $x \in (-1,1)$
Hence f is bijective.

Countable set

Countable set A if $A \sim S$ for some $S \subseteq \mathbb{N}$. Opposite: uncountable.
Example is $\mathbb{Z}$
Lemma:
When A conuntable $\leftrightarrow, \exists f : A \to \mathbb{N}$ injective.
Proof:

Proof part 1	Proof part 2
$S \subseteq \mathbb{N}$	
$f : A \to S$ bijjective	$f : A \to \mathbb{N}$ injective
So $f : A\mathbb{N}$ injective	$S = \text{ran}(f)$
	$f : A \to S$ bijective.

Lemma:
A countable $\leftrightarrow g : \mathbb{N} \to A$ surjective
Proof:

Proof part 1	Proof part 2
$f : A \to S \subset \mathbb{N}$ bijective	take smalles n_a to make it unique.
$\forall n \in S \exists$ unique $a_n \in A$ s.t. $f(a_n) = n$	$\forall a \in A \exists$ smalles $n_a \in \mathbb{N}$ s.t. $g(n_a) = a$
Define $g : \mathbb{N} \to A$	Define $f : A \to \text{ran} f \subset \mathbb{N}$, where $f(a) = n_a$
$g(n) = \begin{cases} a_n \text{ if } n \in S \\ \text{any element in A } if\ n \notin S \end{cases}$	$g(n_a) = a$ and $f(a) = n_a$
The map g is surjective.	The map f is bijective

Corollary:

$$\left.\begin{matrix} B \text{ contable} \\ f : A \to B \text{ injective} \end{matrix}\right\} \Rightarrow A \text{ countable.}$$
$$\left.\begin{matrix} A \text{ contable} \\ g : A \to B \text{ surjective} \end{matrix}\right\} \Rightarrow B \text{ countable.}$$

Theorem: A_n countable for all $n \in \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ countable.

Example:

1:
$\mathbb{N}\times\mathbb{N}=\{(n,m):n,m\in\mathbb{N}\}$ is countable since: $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}, f(n,m)=2^n3^m$ is injective.
EXERCISE: find a bijective map $f\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$
2:
A, B countable $\to A\cup B$ countable.
Assume: $f:A\to\mathbb{N}$ and $g:B\to\mathbb{N}$ injective, and let:
$h:A\cup B\to\mathbb{N}$
$h(x)=\begin{cases}2f(x) \text{ if } x\in A\\ 2g(x)+1 \text{ if } x\in B \text{ and } x\notin A\end{cases}$ This map h is injective.
3:
$A_n=\{0,\pm\frac{1}{n},\pm\frac{2}{n},\ldots\}$ countable.
Why?$\mathbb{Q}=\bigcup_{n=1}^{\infty}A_n$ is countable.

Uncountable sets

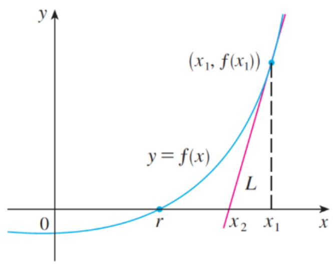
Theorem	The interval $(0,1)$ uncountable	$\mathbb{R}$ uncountable.
Proof	Cantor (1891) Take $g:\mathbb{N}\to(0,1)$ Then: $g(1)=0.d_11d_12d_13d_14\ldots$ $g(2)=0.d_{21}d_{22}d_{23}d_{24}\ldots$ $\vdots$ Define $t\in(0,1)$ by $t=0.c_1c_2c_3c_4\ldots$ Where $c_n=\begin{cases}2 \text{ if } d_{nn}\neq 2\\ 3 \text{ if } d_{nn}=3\end{cases}$ Then $t\neq g(2)$ for all $n\in\mathbb{N}$ So g is not surjective	Assume $\mathbb{R}$ countable If $g:\mathbb{N}\to\mathbb{R}$ surjective then: $\mathbb{R}=\{x_1,x_2,x_3,\ldots,\}$ where $x_n=g(n)$ So we show that $\exists x\in\mathbb{R}$ s.t. $x\neq x_n$ where $n\in\mathbb{N}$ Choose closed and bounded intervals as follows: I_1 s.t. $x_1\notin I_1$ $I_2\subseteq I_1$ s.t. $x_2\notin I_2$ $\vdots$ NIP $\Rightarrow \exists x\in\mathbb{R}$ s.t $x\in\bigcap_{n=1}^{\infty}I_n$ But $x\neq x_n\forall n\in\mathbb{N}$ because $x_n\notin I_n$
Corollarly		$\mathbb{Q}^c=\mathbb{R}\setminus\mathbb{Q}$ $\mathbb{Q}$ countable, $\mathbb{Q}^c$ countable So $\mathbb{Q}\cup\mathbb{Q}^c$ countable, contradiction. There are more irrationals then rationals.

Lecture 4

tangent line, sequence and neighborhood:

NEWTON'S ROOT FINDING METHOD:

Newton's root finding method



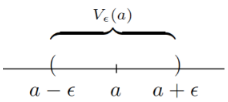
Where equation tangent line: $y = f'(x)(x - x_1) + f(x)$ and:

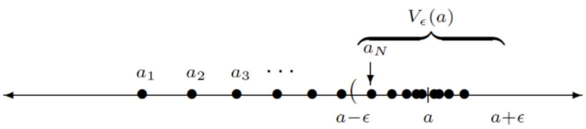
Root of tangent line $x_2 := x_1 - \frac{f(x_1)}{f'(x_1)}$ | Ternative proces $x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 1, 2, \ldots$

SEQUENCE: a function with domain $\mathbb{N}$
Can be written as infinte list of numbers:
(-) $(1, \frac{1}{n}, \frac{1}{3}, \ldots)$
(-) $(\frac{n+1}{n})_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots)$ $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 1)$

LIMIT OF A SEQUENCE: (a_n) converges to a if $\forall \varepsilon > 0$,there $\exists N \in \mathbb{N}$ s.t. $n \geq N \rightarrow |a_n - a| < \varepsilon$
Notation: $a = \lim a_n$ or $a_n \rightarrow a$. So a_n gets arbitrarily close to a as n grows larger.

NEIGHBORHOOD: (1) the set $V_\varepsilon = \{x \in \mathbb{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$ for $a \in \mathbb{R}$—,and $\varepsilon > 0$
NEIGHBORHOOD: (2) $\forall \varepsilon > 0$,there $\exists N \in \mathbb{N}$ s.t. $n \geq N \rightarrow a_n \in V_\varepsilon(a)$ when a_n converges to a
So the tail of the sequence get trapped in $V_\varepsilon(a)$





Example:

$\lim \frac{1}{n} = 0$	$\lim(\frac{6n+7}{3n+1}) = 2$
	$\left\|\frac{6n+7}{3n+1} - 2\right\| = \left\|\frac{6n+1}{3n+1} - \frac{6n+2}{3n+1}\right\| = \frac{5}{3n+1} < \frac{5}{3n}$
Let $\varepsilon > 0$ arbitrary	Let $\varepsilon > 0$ arbitrary
by AP, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$	by AP, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{3}{5}\varepsilon$
$n \geq N \rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ $\rightarrow \left\|\frac{1}{n} - 0\right\| = \frac{1}{n} < \varepsilon$	$n \geq N \rightarrow \left\|\frac{6n+7}{3n+1} - 2\right\| < \frac{5}{3n}$ $\leq \frac{5}{3N} < \varepsilon$

Limit and (di)convergence

STANDARD LIMITS:

Standard limit	condition	standard limit	condition
$\lim \frac{1}{n^\alpha} = 0$	$\alpha > 0$	$\lim c^n = 0$	$-1 < c < 1$
$\lim c^n n^\alpha = 0$	$-1 < c < 1$,$\alpha \in \mathbb{R}$	$\lim \sqrt[n]{c} = 1$	$c > 0$
$\lim \sqrt[n]{n} = 1$		$\lim \frac{n!}{n^n} = 0$	

DIVERGENT SEQUENCE: a sequence that does not converge.
For example: $(a_n) = (-1, 1, -1, 1, \ldots)$ is divergent.

DEFINITION OF CONVERGENCE: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \rightarrow |a_n - a| < \varepsilon$
DEFINITION OF DIVERGENCE: $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}, \exists n \geq N$ s.t. $|a_n - a| \geq \varepsilon$
PROOF:
Choose $\varepsilon = 1$ and $N \in \mathbb{N}$ arbitrary.
Case: $a \geq 0$ $n = 2N + 1 \rightarrow |a_n - a| = |-1 - a| - 1 + a \geq \varepsilon$
Case: $a < 0$ $n = 2N \rightarrow |a_n - a| = |1 - a| = 1 - a > \varepsilon$

Bounded Sequences:

BOUNDED SEQUENCE (a_n): if $\exists M > 0$ s.t $|a_n| \leq M \forall n \in \mathbb{N}$

Theorem: (a_n) convergent $\rightarrow$ (a_n) bounded.
Note: can be used to prove sequence diverges.
PROOF:
Let $a = \lim a_n$ then for $\varepsilon = 1$ exists $n \in \mathbb{N}$ s.t.: by triangle inequality:
$n \geq N \rightarrow |a_n| - a < 1$ so $||a_n| - |a|| < 1$ so $|a_n| - |a| < 1$ so $|a_n| < 1 + |a|$
For $M = \max\{|a_1|, |a_n|, \ldots, |a_{N-1}|, 1 + |a|\}$ we have $|a_n| \leq M$ for all $n \in \mathbb{N}$
So (a_n) is convergent leads to (a_n) is bounded.

Examples:

1: $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is bounded (take $M = 1$)
2: $(b_n) = (1, 4, 9, 16, 25, \ldots)$ is not bounded.
3: $(a_n) = n^2$ diverges because it is not bounded.
For $M = \max\{|a_1|, |a_n|, \ldots, |a_{N-1}|, 1 + |a|\}$ we have: $|a_n| \leq M$ for all $n \in \mathbb{N}$

Algebraic porperties:

if $a = \lim a_n$ and $b = \lim b_n$ then:

Algebraic propertie	Proof
$\lim(ca_n) = ca$ (where $c \in \mathbb{R}$)	
$\lim(a_n + b_n)a + b$	$\|(a_n + b_n) - (a + b)\| = \|(a_n - a) + (b_n - b)\| \leq \|a_n - a\| + \|b_n - b\|$ Let $\varepsilon > 0$ arbitrary: $\exists N_1 \in \mathbb{N}$ s.t. $n \geq N_1 \to \|a_n - a\| < \frac{1}{2}\varepsilon$ $\exists N_2 \in \mathbb{N}$ s.t. $n \geq N_2 \to \|b_n - b\| < \frac{1}{2}\varepsilon$ $N = \max\{N_1, N_2\}$ then: $n \geq N \to \|(a_n + b_n) - (a + b)\| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$
$\lim(a_n b_n) = ab$	$\|a_n b_n - ab\| = \|a_n b_n - ab_n + ab_n - ab\|$ $= \|b_n(a_n - a) + a(b_n - b)\| \leq \|b_n(a_n - a)\| + \|a(b)n - b\|$ $= \|b_n\|\|a_n - a\| + \|a\|\|b_n - b\| \leq M\|a_n - a\| + \|a\|\|b_n - b\|$ (b_n) convergent and by that bounded. $\varepsilon > 0$ gives: $\exists N_1 \in \mathbb{N}$ s.t. $n \geq N_1 \to \|a_n - a\| < \frac{\varepsilon}{2M}$ $\exists N_2 \in \mathbb{N}$ s.t. $n \geq N_2 \to \|b_n - b\| < \frac{\varepsilon}{2\|a\|}$ Define $N = \max\{N_1, N_2\}$ then: $n \geq N \to \|a_n b_n - ab\| < \varepsilon$
$\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ if $b \neq 0$	

Order properties:

$\lim a_n = a \& \lim b_n = b$ then

Order property	Proof
(1) $a_n \geq 0 \forall n \in \mathbb{N} \to a \geq 0$	assume $a < 0$, for $\varepsilon = \|a\|$ exists $N \in \mathbb{N}$ s.t. $n \geq N \to \|a_n - a\| < \varepsilon \to a - \varepsilon < a_n < a + \varepsilon$ $a_n < a + \varepsilon = 0$ contradiction.
(2) $a_n \leq b_n \forall n \in \mathbb{N} \to a \leq b$	$a_n \leq b_n$ then $b_n - a_n \geq 0$ $b - a = \lim(b_n - a_n) \geq 0 \to b \geq 0$
(3) $c \leq b_n \forall n \in \mathbb{N} \to c \leq b$	$a_n = c$ from 2
(4) $a)n \leq c \forall n \in \mathbb{N} \to a \leq c$	$b_n = c$ from 2

Strict inequalities are not aways preserved.
$\forall n \in \mathbb{N} \frac{1}{n} > 0$ but $\lim \frac{1}{n} = 0$
$\forall n \in \mathbb{N} \frac{n}{n+1} < 1$ but $\lim \frac{n}{n+1} = 1$

Lecture 5

monotone sequence:

MONOTONE SEQUENCE a_n if it is $\begin{cases} \text{increasing: } a_n \leq a_{n+1} \, \forall n \in \mathbb{N} \\ \text{Decreasing: } a_n \geq a_{n+1} \, \forall n \in \mathbb{N} \end{cases}$

(a_n) bounded& monotone $\rightarrow$ (a_n) converges.
PROOF: $A = \{a_n : n \in \mathbb{N}\}$ bounded.
(1) (a_n) increasing $\rightarrow \lim a_n = \sup A$
Proof (CTD) assume (a_n) increases and let $s = \sup\{a_n : n \in \mathbb{N}\}$
Let $\varepsilon > 0$ aribtrary $\rightarrow$ $s - \varepsilon$ not upper bound.
Exists $N \in \mathbb{N}$ s.t. $s = \varepsilon < a_n$. For $N \geq N$ we have:
$s - \varepsilon < a_N \leq a_n \leq s \leq s_\varepsilon \rightarrow |a_n - s| < \varepsilon$ so a_n converges.
(2) (a_n) decreasing $\rightarrow \lim a_n = \inf A$ (exercise!)

Examples:

1: $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ and $(b_n) = (1, 1, 2, 2, 4, 4, \ldots)$ are monotone.
2: $(c_n) = (1, 0, 1, 0, \ldots)$ is not monotone.
3: if $a_{n+1} = \sqrt{1 + a_n}$ with $a_1 = 1$ then (a_n) converges.
(a) proof by induction that a_n is increasing.
Base case:
$a_1 = 1, a_2 = \sqrt{2}$ so $a_1 < a_2$
Induction step:
Assume $a_n < a_{n+1}$ for some n we have: $1 + a_n < 1 + a_{n+1} \rightarrow \sqrt{1 + a_n} < \sqrt{1 + a_{n+1}} \rightarrow a_{n+1} < a_{n+2}$
So $a_n < a_{n+1} < a_{n+2} < \ldots$ so increasing.
(b) proof by induction that (a_n) is bounded.
$a_1 = 1 \rightarrow a_1 < 2$
$a_n < 2$ for some $n \rightarrow 1 + a_n < 3 \rightarrow \sqrt{1 + a_n} < \sqrt{3} < \sqrt{2} \rightarrow a_{n+1} < 2$
So a bounded sequence.
(c) Find $\lim a_n$
By MCT, exists $a = \lim a_n$ $a_{n+1}^2 = 1 + a_n$ so $\lim a_{n+1}^2 = \lim(1 + a_n) \Rightarrow a^2 = 1 + a \Rightarrow a = \frac{1+\sqrt{5}}{2}$

Subsequences:

Pick $n_k \in \mathbb{N}$ s.t.: $1 \leq n_1 < n_2 < n_3 < \ldots$
If (a_n) is a sequence then: $(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$ is called A SUBSEQUENCE OF (a_n)
Note: $n_k \geq k$ for all $k \in \mathbb{N}$

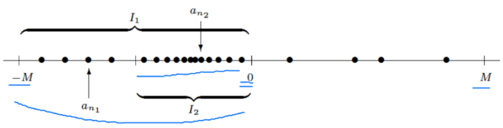
Theorem: $\lim a_n = a \to \lim a_{n_k} = a$
PROOF:
Let $\varepsilon > 0$ arbitrary so $\exists N \in \mathbb{N}$ s.t $n \geq N \to |a_n - a| < \varepsilon$
Use $n_k \geq k$ so you can say that $k \geq N \Rightarrow n_k \geq N$
So $|a_{n_k} - a| < \varepsilon$

Examples:

1:
$(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$
Example of subsequences:
$n_k = k + 4 \to (a_{n_k}) = (\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots)$
$n_k = 2k \to (a_{n_k}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$
$n_k = 10^k \to (a_{n_k}) = (\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots)$ **2:**
$(a_n) = (-1, 1, -1, 1, \ldots)$ diverges:
Take 2 subsequences:
$n_k = 2k \to (a_{n_k}) = (1, 1, 1, 1, \ldots) \to \lim a_{n_k} = 1$ $n_k = 2k - 1 \to (a_{n_k}) = (-1, -1, -1, -1, \ldots) \to \lim a_{n_k} = -1$
Different subsequences have different limits $\to (a_n)$ diverges.

Bolzano-Weierstrass theorem:

Every bounded sequence convergent subsequence
PROOF:
$\forall n \exists M > 0$ s.t. $a_n \in [-M, M]$
Every bounded sequence has a convergent subsequence.



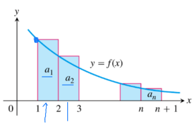
Halving proces: nested intervals: $I_1 \subset I_2 \subset I_3 \subset \cdots \Rightarrow$ NIP $\to$ there exists $x \in \bigcap_{n=1}^{\infty} I_n$
Each I_k contains infinitely many terms of sequence.
Pick $n_1 \in \mathbb{N}$ with $a_{n_1} \in I_1$
Pick $n_2 \in \mathbb{N}$ with $n_2 > n_1$ and $a_{n_2} \in I_2$
Pick $n_3 \in \mathbb{N}$ with $n_3 > n_2$ and $a_{n_3} \in I_3$
$\vdots$
Note that $\left.\begin{matrix} x \in I_k \\ a_{n_k} \in I_k \end{matrix}\right\} \to |a_{n_k} - x| \leq \text{length}(I_k) = \frac{2M}{2^k} \to 0$
So convergent subsequence.

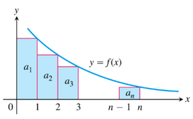
add infinitely many numbers.

infinite series: $\sum\limits_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$
$n-$th partial sum: $s_n = a_1 + a_2 + \ldots + a_n$
if $s_n = s$ then we say that the series converges to s
EULER'S FAMOUS EXAMPLE:
$\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converges:
PROOF:
$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}$ so $s_n < s_{n+1}$ for all $n \in \mathbb{N}$ so $s_n < 2$ for all $n \in \mathbb{N}$
MCT: Limits s_n exists.
Why is $s_n < 2$ for all $n \in \mathbb{N}$?
$s_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \ldots + \frac{1}{n\cdot n} < 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \ldots + \frac{1}{n(n-1)}$
$= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n-1} - \frac{1}{n} = 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n}$
$s_n < 2 - \frac{1}{n}$ so $s_n < 2$
Remark: since $s_n < 2$,for all n the order limit theorem implies:
$\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \lim s_n \leq 2$
Euler found also: $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and $\sum\limits_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$
For even power of k we know the solution of the infinite summ, for odd powers of k the solution is unknown.

The harmonic series and intergral test for converges:

Harmonic series: $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges.
PROOF:
$s_n = 1 + \frac{1}{n} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$
$s_{n^k} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \ldots + \frac{1}{8}) + \ldots + (\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k})$
$s_{n^k} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + \ldots + (\frac{1}{8} + \ldots + \frac{1}{8}) + \ldots + (\frac{1}{2^k} + \ldots + \frac{1}{2^k})$
$= 1 + \frac{1}{2} + 2(\frac{1}{4} + 4(\frac{1}{8}) + \ldots + 2^{k-1}(\frac{1}{2^k})$
$s > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}$
$s > 1 + \frac{k}{2}$ for all $k \in \mathbb{N}$
So s_n is unbounded (because the subsequence is divergent) and therefore s_n is divergent.
The integral test:
Assume that $f : [1, \infty) \to \mathbb{R}$ is positive, continuous and monotonically decreasing.
Let $a_k = f(k)$ then $\sum\limits_{k=1}^{\infty} a_k$ converges $\leftrightarrow \int\limits_{k=1}^{\infty} f(x)dx < \infty$
PROOF: where $s_n = a_1 + a_2 + \ldots + a_n$ because $a_k > 0$ increasing.





So $\int\limits_{1}^{n+1} f(x)dx \leq s_n \leq a_1 + \int\limits_{1}^{n} f(x)dx$ for all $n \in \mathbb{N}$

$\int\limits_{1}^{\infty} f(x)dx < \infty$so s_n bounded& convergent, $\int\limits_{1}^{\infty} f(x)dx = \infty$so s_n unbounded& divergent.

Lecture 6

Cauchy sequence:

Name	Theorem	Proof or meaning.
Cauchy Sequence	$\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $n, m \geq N \to \lvert a_n - a_m \rvert < \varepsilon$	The terms get close to eachother
	(a_n) convergent $\to$ (a_n) cauchy	assume $a = \lim a_n$ For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \to \lvert a_n - a \rvert < \frac{1}{2}\varepsilon$ $m, n \geq N \to \lvert a_n - a_m \rvert = \lvert (a_n - a) - (a_m - a) \rvert$ $\leq \lvert a_n - a \rvert + \lvert a_m - a \rvert < \varepsilon$
Lemma	(a_n) cauchy $\to$ (a_n) bounded	For $\varepsilon = 1$ there exists $N \in \mathbb{N}$ s.t. $n, m \geq N \to \lvert a_n - a_m \rvert < 1$ fix $m = N$: $n \geq N \to \lvert a_n - a_N \rvert < 1$ $\to \lvert a_n - \lvert a_N \rvert \rvert < 1$ $\to \lvert a_n \rvert - \lvert a_N \rvert < 1$ $\to \lvert a_n \rvert < 1 + \lvert a_N \rvert$ For $M = \max\{\lvert a_1 \rvert, \lvert a_2 \rvert, \dots, \lvert a_{n-1}, 1 + \lvert a_N \rvert \rvert\}$ we have $\lvert a_n \rvert \leq M$ for all $n \in \mathbb{N}$
	(a_n) Cauchy $\to$ (a_n) convergent	Lemma gives (a_n) bounded. BW gives (a_n) convergent subsequence (a_{n_k}) so $a = \lim(a_{n_k})$ for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $n, m \geq N \to \lvert a_n - a_m \rvert < \frac{1}{2}\varepsilon$ Fix an index: $n_k > N$ s.t. $\lvert a_{n_k} - a \rvert < \frac{1}{2}\varepsilon$, then: $n \geq N \to \lvert a_n - a \rvert = \lvert a_n - a_{n_k} + a_{n_k} - a \rvert$ $\lvert a_n - a \rvert \leq \lvert a_n - a_{n_k} \rvert + \lvert a_{n_k} - a \rvert$ $\lvert a_n - a \rvert < \varepsilon$

Properties of series and algebraic limit theorem:

Infinite series: $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$
n-th partial sum: $s_n = a_1 + a_2 + \dots + a_n$
Convergence: $\sum_{k=1}^{\infty} a_k = A \leftarrow$ by definition $\to \lim s_n = A$

Algebraic limit theorem:
if $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ then:
(1) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$
(2) $\sum_{k=1}^{\infty} (a_k + b_k)A + B$
Proof:
Apply analogous theorem for sequences to partial sums.

Cauchy criterion:

Theorem: The following statements are equivalent:
(1) $\sum_{k=1}^{\infty} a_k$ converges.
(2) for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $n > m \geq N \rightarrow |a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon$
PROOF:
Note that: $|s_n - s_m| = |a_{m+1} + \ldots + a_n|$
Statement 1$\Leftrightarrow$ (s_n) converges $\Leftrightarrow$ (s_n) Cauchy $\Leftrightarrow$ statement 2.
So equivalent.

Example:

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
For any $m \in \mathbb{N}$ and $n = 2m$ we have:
$|a_{m+1} + a_{m+2} + \ldots + a_n| = \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} > \frac{m}{2m} = \frac{1}{2}$
So: $|a_{m+1} + a_{m+2} + \ldots + a_n| > \frac{1}{2}$
Hence, the Cauchy criterion fails. So, this serie is diverges.

Necessary condition for convergence:

Theorem: $\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow \lim a_k = 0$
PROOF:
Let $\varepsilon > 0$ be arbitrary.
There exists $N \in \mathbb{N}$ s.t. $n > m \geq N \Rightarrow |a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon$
$n = m + 1$ and $m \geq N \Rightarrow |a_{m+1}| < \varepsilon$
Warning: opposite is not true. Counterexample: $\lim \frac{1}{k} = 0$ but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
Note:
The previous theorem also gives a test for divergence.
Example: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{2k} = 1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \ldots$.
Diverges since $\lim a_k = \lim (-1)^{k+1} \cdot \frac{k+1}{2k}$ does not exist.

Comparison test

Theorem if $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$, then:
(1) $\sum_{k=1}^{\infty} b_k$ converges $\rightarrow \sum_{k=1}^{\infty} a_k$ converges.
(2) $\sum_{k=2}^{\infty} a_k$ diverges $\rightarrow \sum_{k=2}^{\infty} b_k$ diverges
PROOF:
$|a_{m+1} + a_{m+2} + \ldots + a_n| = a_{m+1} + a_{m+2} + \ldots + a_n$
$\leq b_{m+1} + b_{m+2} + \ldots + b_n = |b_{m+1} + b_{m+2} + \ldots + b_n|$
Apply the cauchy criterion for series.
Note:
Theorem does not have to hold for all k but just for large k

Example:
$\sum_{k=1}^{\infty} \frac{1}{k!}$ converges
For $k \geq 4$ we have: $k! \geq k^2 \to \frac{1}{k!} \leq \frac{1}{k^2}$
Apply comparison test: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $\to \sum_{k=1}^{\infty} \frac{1}{k!}$ converges.

Alternating series test:

Theorem: assume:
(-) $0 \leq a_{k+1} \leq a_k$ for all $k \in \mathbb{N}$
(-) $\lim a_k = 0$
Then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.
PROOF:
Consider the partial sums:
$s_n = a_1 - a_2 + a_3 - \ldots + (-1)^{n+1} a_n$
Proof (Ctd): the partial sums form nested intervals:
$I_n = [s_{2n}, s_{2n-1}] \Rightarrow I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$
NIP $\Rightarrow \exists s \in \mathbb{R}$ s.t. $s \in I_n$ forall $n \in \mathbb{N}$
let $\varepsilon > 0$ be arbitrary.
Choose $N \in \mathbb{N}$ s.t. $a_{2N} < \varepsilon$ then:
$n \geq 2N \Rightarrow s, s_n \in I_n = [s_{2N}, s_{2n-1}]$
$\Rightarrow |s - s_n| \leq s_{2N-1} - s_{2N}$
$\Rightarrow |s - s_n| \leq a_{2N}$
$\Rightarrow |s - s_n| < \varepsilon$

Example:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} \ldots$ converges.
This follows from the alternating series test:
$a_k = \frac{1}{k}$ satisfies $0 \leq a_{k+1} \leq a_k$ and $\lim a_k = 0$

Absolute vs. conditional convergence:

Theorem: $\sum_{k=1}^{\infty} |a_k|$ converges $\to \sum_{k=1}^{\infty} a_k$ converges.
PROOF:
$0 \leq a_k + |a_k| \leq 2|a_k|$ for all $k \in \mathbb{N}$
Comparison test $\to \sum_{k=1}^{\infty} (a_k + |a_k|)$ converges.
Apply Algebraic limit theorem:
$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k|$ converges.

Absolute and conditional convergent:

$\sum_{k=1}^{\infty} a_k$ is called:

(1) ABSOLUTELY CONVERGENT if $\sum_{k=1}^{\infty} |a_k|$ converges. Example: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

(2) CONDITIONALLY CONVERGENT if it converges, but $\sum_{k=1}^{\infty} |a_k|$ diverges. Example: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

Geometric and telescoping series:

GEOMETRIC SERIES: is of the form: $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \ldots$

PARTIAL SUMS: $s_n = a + ar + ar^2 + \ldots + ar^{n-1} \Rightarrow rs_n = ar + ar^2 + ar^3 + \ldots + ar^n \Rightarrow (1-r)s_n = a(1-r^n)$

For $|r| < 1$ we have: $s_n = \lim \frac{(1-r^n)}{1-r} = \frac{a}{1-r}$

TELESCOPING SERIES: of the form $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (b_k - b_{k+1})$

Successive terms cancel eachother out:

$s_n = a_1 + a_2 + a_3 + \ldots + a_n$

$s_n = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots + (b_n - b_{n+1}) = b_1 - b_{n+1}$

The series converges $\Leftrightarrow$ (b_n) converges.

Example:

1:

We have $0.999\ldots = 1$

This follows from:

$0.999\ldots = \sum_{k=1}^{\infty} \frac{9}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} 9(\frac{1}{10})^k = \frac{1}{10} \cdot \frac{9}{1-\frac{1}{10}} = 1$

2:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots = 1$$

Solution:

$$s_n = \sum_{k=1}^{n} (\frac{1}{k} - \frac{1}{k+1})$$
$$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n+1})$$
$$= 1 - \frac{1}{n+1} \to 1$$

3:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12} = \frac{1}{4}$$

Solution

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2+7k+12} = \sum_{k=1}^{n} \frac{1}{(k+3)(k+4)} = \sum_{k=1}^{n} (\frac{1}{k+3} - \frac{1}{k+4})$$
$$= (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{7}) + \ldots + (\frac{1}{n+3} - \frac{1}{n+4})$$
$$= \frac{1}{4} - \frac{1}{n+4} \to \frac{1}{4}$$

Lecture 7

open and closed intervals, open sets:

Closed interval: (endpoints included): $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
Open interval: (endpoints not included): $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
How to define open and closed for arbitrary sets?
Open sets: $O \subset \mathbb{R}$ open if $\forall a \in O$ there $\exists \varepsilon > O$ s.t. $V_\varepsilon \subset O$
Recall: $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$
Note: the empty set $\emptyset$ is open by definition.

Example:

1:
the interval (c, d) is open. take $x \in (c, d)$ arbitrary.
Take $\varepsilon = \min\{|x - c|, |x - d|\}$,then $V_\varepsilon \subset (c, d)$
2:
The interval $[c, d)$ is not open, for $x = c$ no $\varepsilon > 0$ works.
Because for any ε ,$c - \varepsilon$ is not in the interval.
3:
$\mathbb{Q}$ is not open.
Take $\varepsilon > 0$ arbitrary.
Take $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{e}{\sqrt{2}}$ and set $x = \frac{\sqrt{2}}{n}$
Then $x \in V_\varepsilon(0)$ but $x \neq \mathbb{Q}$

Unions and intersections:

Theorem:
(1) Union of arbitrary collections of open sets are open.
(2) Intersections of finite collections of open sets are open.
Proof:
(1) Let $O = \bigcup_{i \in I} O_i$ with each O_i open.
$x \in O \to x \in O_i$ for some $i \in I$
There exists $\varepsilon > 0$ s.t. $V_\varepsilon(X) \subseteq O_i \subseteq O$
(2) let $O = O_1 \cap O_2 \cap \ldots \cap O_n$ with each O_i open.
$x \in O \to x \in O_i$ for all $i = 1, \ldots, n$
For all $i = 1, \ldots, n$ there exists, $\varepsilon_i > 0$ such that $V_{\varepsilon_i}(x) \subseteq O_i$
For $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ we have: $V_\varepsilon(x) \subseteq O_i$ for all $i = 1, \ldots, n$
Warning: intersection infinitely many open sets need not to be open: Counterexample: O_n is open for all $n \in \mathbb{N}$: because $\bigcap_{n=1}^{\infty} O_n = \{0\}$ is not open.

Warning:

The intersection of infinitely many open sets NEED NOT BE open!
Counterexample: $)_n = (-\frac{1}{n}, \frac{1}{n})$, is open for all $n \in \mathbb{N}$
$\bigcap_{n=1}^{\infty} O_n = \{0\}$ is not open!

Limit points:

LIMIT POINT: x is a limit point of $A \subseteq \mathbb{R}$ if:
$\forall \varepsilon > 0$ of $V_\varepsilon(x)$ intersects A in some point other than x
Note: limit points of A may or may not belong to A.
Theorem: The following statements are equivalent.
(1) x is a limit point of A
(2) There exists a sequence $a_n \neq x$,$\forall n \in \mathbb{N}$ and $x = \lim a_n$
PROOF:
$1 \to 2$
Let $n \in \mathbb{N}$ and set $\varepsilon = \frac{1}{n}$
There exists $a_n \in V_\varepsilon(x) \cap A$ with $a_n \neq x$
Note that: $|a_n - x| < \varepsilon = \frac{1}{n}$
$2 \to 1$
for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t.:
$n \geq N \to |a_n - x| < \varepsilon$
By assumption $A_N \neq x$ and $A_n \in A$ we can conclude that $A_n \in V_\varepsilon(x)$

Example:

1:	**2:**
$x = 0$ is a limit point of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$	$x = 0$ and $x = 1$ are limits of $A = (0, 1)$
Take $\varepsilon > 0$ arbitrary.	For $x = 0$ take $a_n = \frac{1}{2n}$
Take $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$	For $x = 1$ take $a_n = \frac{n}{n+1}$
Then $\frac{1}{n} \in V_\varepsilon()) \cap A$	
Note: $0 \notin A$	

Prove same result by means of definition.

Closed sets:

CLOSED TEST: contains it limits. Can't leave set by taking limits.
Theorem: Equivalent:
(1) F is closed
(2) Every Cauchy sequence in F has its limit in F
PROOF:
$1 \to 2$ Let $(a_n) \subset F$ be Cauchy.
$x = \lim a_n$ exists; now consider 2 cases:
(a): $x \neq a_n$ then for all $n \in \mathbb{N} \to x$ is a limit point of $F \to x \in F$
(b): $x = a_n$ for some $n \in \mathbb{N} \to x \in F$ holds trivially.
$2 \to 1$ Let x be a limit point of F
$x = \lim a_n$ with $a_n \in F$ and $a_n \neq x$ for all $n \in \mathbb{N}$
(a_n) convergent $\to (a_n)$ Cauchy $\to x \in F$ by assumption.

Example:

$[c, d]$ is closed.
Let x be a limit point of $[c, d]$ $x = \lim x_n$ for some sequence $(x_n) \subseteq [c, d]$
$c \leq x_n \leq d$ for all $n \in \mathbb{N}$
Order limit theorem: $c \leq x \leq d \to x \in [c, d]$

Closure:

CLOSURE OF A: $\overline{A} = A \cup \{\text{all limit points of } A\}$
Theorem: $\overline{A}$ is closed.
PROOF:
(1) x limit point of A and $A \subset \overline{A}$ then x limit point $\overline{A}$
(2) $A = A \cup LL$ with $L = \{\text{Limit points of } A\}$
x limit point of $\overline{A} \rightarrow \forall \varepsilon > 0$ there $\exists y \in V_\varepsilon(x) \cap \overline{A}$ where $y \neq x$
So $y \in A \vee y \in L$
(a) $y \in A \rightarrow x$ is a limit point of A
(b) $y \in L$
$\rightarrow \forall \delta > 0$ there $\exists z \in V_\delta(y) \cap A$ where $z \neq y$
Note: $V_\delta(y) \subset V_\varepsilon(x)$ around $\{x\}$ for δ small enough
$\rightarrow x$ is a limit point of A
Theorem completeness:
(1) O open $\Leftrightarrow O^c$ closed.
(2) F closed $\Leftrightarrow F^c$ open.

MUTUALLY EXCLUSIVE:
Sets are not open OR closed. They can be neither open nor closed $(0,1]$ and $\mathbb{Q}$, but they also can be open and closed, $\mathbb{R}$ and $\emptyset$
So impossible to prove openess or closeness by contradiction.

UNIONS AND INTERSECTIONS:
(1) uninons of finite collections of closed sets are closed.
(2) intersections of arbitrary collections of closed sets are closed.
PROOF:

(1)
$F_1, \ldots, F_n$ closed
$F_1^c, \ldots, F_n^c$ open $\rightarrow F_1^c \cap \ldots \cap F_n^c$ open
$\rightarrow (F_1^c \cap \ldots \cap F_n^c)^c$ closed $\rightarrow F_1 \cup \ldots \cup F_n$ closed.

(2)
F_i^c open for all $i \in I$
$\bigcup_{i \in I} (F_i)^c$ open $\rightarrow (\bigcup_{i \in I} F_i^c)^c$ closed
$\rightarrow \bigcap_{i \in I} F_i$ closed.

Warning: union infinitely many closed sets need not to be closed.
Counterexample: $F_n = [-\frac{n}{n+1}, \frac{n}{n+1}]$ closed for all $n \in \mathbb{N}$ but $\bigcup_{n=1}^{\infty} F_n = (-1,1)$ not closed.

example

1:
ïf $A = (0,1)$ then $\overline{A} = [0,1]$
All points of A are limit points.
Also, $x = 0$ and $x = 1$ are limit points.
If $x < 0$ or $x > 1$ then x is not a limit point of A

2:
$\overline{\mathbb{Q}} = \mathbb{R}$
Take $x \in \mathbb{R}$ and $\varepsilon > 0$ arbitrary.
$\mathbb{Q}$ is dense in $\mathbb{R}$: there exists $r \in \mathbb{Q}$
such that $x < r < x + \varepsilon$
Hence $\in V_\varepsilon(x) \cap \mathbb{Q}$ and $r \neq x$
So, each $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$

Lecture 8

Sequential definition:

Compact set a set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a convergent subsequence with a limit in K

Theorem:
$K \subseteq \mathbb{R}$ compact $\leftrightarrow$ K closed and bounded.
Proof:

$\rightarrow$	$\leftarrow$
Assume k not bounded	$(x_n) \subseteq K$
exists $x_n \subseteq K$ with $\lvert x_n \rvert > n$ for all $n \in \mathbb{N}$	K bounded, so (x_n) bounded.
x_n no convergent subsequence.	B-w theorem: (x_n) convergent subsequence.
Contradiction: K bounded.	$x = \lim x_{n_k}$
	K closed $\rightarrow x \in K$
x limit point of K ,prove $x \in K$	
$\exists x_n \subseteq K$ s.t. $x = \lim x_n$	
K compact $\exists (x_{n_k})$ converge to y where $y \in K$	
$(x_{n_k}) \rightarrow x$ as well $x = y \in K$	

Generalization of NIP:
Theorem: Assume $K_n \neq \emptyset$ is compact for all $n \in \mathbb{N}$ and $K_1 \supseteq K_2 \supseteq \ldots$ then $\bigcap_{n=1}^{\infty} K_n$ nonempty.

Example:

1:	**2:**
Every finite set is compact	$[a, b]$ compact
Let $K = \{a_1, a_2, \ldots, a_p\}$	Let $(x_n) \subseteq [a, b]$ arbitrary
Let $(x_n) \subset K$ be arbitrary.	(x_n) bounded.
Without loss of generality $x_n = a_1$	BW-theorem: (x_n) convergent subsequence (x_{n_k})
for infinitely many $n \in \mathbb{N}$	Let $x = \lim x_{n_k}$
Take (x_{n_k}) s.t. $x_{n_k} = a_1$ for all $k \in \mathbb{N}$	Order limit theorem: $a \leq x_{n_k} \leq b$ for all k
$\lim x_{n_k} = a_1 \in K$	$a \leq x \leq b$
3:	**4:**
$(0, 1]$ not compact	$\mathbb{R}$ not compact
Take $x_n = \frac{1}{n} \in (0, 1]$	$x_n = n$ no convergent subsequence.
Every subsequence $(x_{n_k}$ has	
$\lim x_{n_k} = 0$ but $0 \notin (0, 1]$	
5	**6**
Every finite set compact	$K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ not compact
$K = \{a_1, a_2, \ldots, a_p\}$	K bounded: $\lvert x \rvert \leq 1$ for each $x \in K$
K bounded: $x \in K \rightarrow$	K closed if $x < 0$ or $x > 0$ then
$\lvert x \rvert \leq M = \max\{\lvert a_1 \rvert, \ldots \lvert a_p \rvert\}$	x not limit point of K (exercise!)
K closed: $a_1 < a_2 < \ldots < a_p$	$x = 0$ limit point of K and, $x \in K$
$K^c = (-\infty, a_1) \cup (a_1, a_2) \cup \ldots \cup (a_p, \infty)$ open	

Open covers:

$A \subseteq \mathbb{R}$ and assume $O_i \subseteq \mathbb{R}$ where $i \in I$ are open.
OPEN COVER: O_i if $A \subseteq \bigcup_{i \in I} O_i$

Theorem: K compact $\leftrightarrow$ K has a finite subcover.
PROOF:

$\Leftarrow$	$\Rightarrow$
$O_n = (-n, n)$,$n \in \mathbb{N}$ open cover K	$O_i, i \in I$ open cover K without finite subcover
$K \subset O_1 \cup \ldots \cup O_N = (-N, N)$ for some $N \in \mathbb{N}$.	Take bounded closed interval $J_1 \subseteq K$
Therefore, K is bounded.	Halving proces: construct J_n s.t.:
	$J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots$
	$K \cap J_n$ not be convered by finitely many O_i's
	$K \cap J_n$ compact for all $n \in \mathbb{N}$
	Length $J_n = \frac{J_1}{2^{n-1}} \to 0$
Let y be a limit point of K	$\bigcap_{n=1}^{\infty} (K \cap J_n) \neq \emptyset$
There exists $(y_n) \subset K$ with $y = \lim y_n$.	$\exists x \in K$ s.t. $x \in J_n$ for all n
Assume $y \notin K$ Let $x \in K$ and $O_x = V_\varepsilon(x)$	$x \in O_i$ for $i \in I$ and $\varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq O_i$
$\varepsilon = \frac{1}{2}\|x - y\|$	$\exists N \in \mathbb{N}$ s.t.length $(J_N) < \varepsilon$
Set O_x open cover K	Hence $K \cap J_N \subseteq V_\varepsilon(x) \subseteq O_i$ contradiction.
$\exists x_1, \ldots, x_2 \in K$ s.t. $K \subseteq O_{x_1} \cup \ldots \cup O_{x_n}$	
Pick $N \in \mathbb{N}$ s.t. $\|y_N - y\| < \min\{\frac{1}{2}\|x_i - y\| : i = 1, \ldots, n\}$	
Hence $y_n \notin O_{x_1} \cup \ldots \cup O_{x_n}$ contradiction	

Heine Borel theorem: Let $K \subseteq \mathbb{R}$ then following statements equivalent:
(1) K is compact
(2) K is closed and bounded.
(3) Any open cover K has a finite subcover.

Example:

1:
Possible open covers for $A = (0, 1)$:
$O_1 = \mathbb{R}$
$O_1 = (0, 1)$
$O_1 = (0, \frac{1}{2})$ and $O_2 = (\frac{1}{3}, 5)$
$O_2 = (-\frac{n}{10}, \frac{n}{10})$,$n \in \mathbb{N}$. Has a finite subcover! $O_a = (\frac{1}{a}, 2)$,$a \geq 1$ does not have a finite subcover!
2:
Every finite set is compact:
Let $K = \{a_1, a_2, \ldots, a_p\}$
Let O_i where $i \in I$ be an open cover for K
There exists $i_1, \ldots i_p \in I$ s.t. $a_k \in O_{i_k}$
Therefore $K \subset O_{i_1} \cup \ldots \cup O_{i_p}$

Lecture 9

LIMIT POINT: c is a limit point of A where $f : A \to \mathbb{R}$ when:
$\lim\limits_{x \to c} f(x) = L$ when: $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\begin{Bmatrix} 0 < |x-c| < \delta \\ x \in A \end{Bmatrix} \Rightarrow |f(x) - L| < \varepsilon$
Note: f need not be defined at c
Note: type definition: ε, δ definition.

SEQUENTIAL CHARACTERIZATION:
Let $f : A \to \mathbb{R}$ and c a limit point of A the following statements are equivalent:
(1) $\lim\limits_{x \to c} f(x) = L$
(2) $\lim f(x_n) = L$ for all $(x_n) \subset A$ with $x_n \neq c$ and $\lim x_n = c$
(3) $\lim\limits_{x \to c} f(x)$ does not exist if there exist $(x_n), (y_n) \subseteq A$ s.t.
(a) $x_n \neq c$ and $y_n \neq c$
(b) $\lim x_n = \lim y_n = c$
(c) $\lim f(x_n) \neq \lim f(y_n)$

Example:

1:
$\lim\limits_{x \to 2} \frac{x^2+x-6}{5x-10} = 1$
Let $\varepsilon > 0$ be arbitrary and set $\delta = 5\varepsilon$
If $0 < |x-2| < \delta$,then:
$\left| \frac{x^2+x-6}{5x-10} - 1 \right| = \left| \frac{(x+3)(x-2)}{5(x-2)} - 1 \right| = \left| \frac{x+3}{5} - 1 \right| = \frac{|x-2|}{5} < \frac{5}{\delta} = \epsilon$
2:
$\lim\limits_{x \to c} \sqrt{x} = \sqrt{c}$ for $c > 0$
$|\sqrt{x} - \sqrt{c}| = \left| \frac{x-c}{\sqrt{x}+\sqrt{c}} \right| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}}$
With $\varepsilon > 0$ and $\delta = \sqrt{c} \cdot \varepsilon$ the definition is satisfied.
So, $|\sqrt{x} - \sqrt{c}| \leq \frac{|x-c|}{\sqrt{c}}$
3:
$\lim\limits_{x \to 0} f(x)$ does not exist for:
$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ and take $x_n = \frac{1}{n}$ and $y_n = \frac{\sqrt{2}}{n}$ then it satify:
$\lim x_n = \lim y_n = 0$
$\lim f(x_n) = 1$ and $\lim f(y_n) = 0$ so the limit does not exist.

Algebraic porperties:

Let $f, g : A \to \mathbb{R}$,c a limit point of A and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ Then:

Algebraic property	condition	Algebraic property
(1) $\lim_{x\to c} kf(x) = kL$	$k \in \mathbb{R}$	$\lim_{x\to c} f(x) + g(x) = L + M$
$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$	$M \neq 0$	$\lim_{x\to c} f(x)g(x) = LM$

CONTINOUS function $f : A \to \mathbb{R}$ if $\forall\varepsilon > 0$ there $\exists\delta > 0$ s.t. $\left.\begin{cases} |x-c| < \delta \\ x \in A \end{cases}\right\} \Rightarrow |f(x) - f(c)| < \varepsilon$

Notes:
(1) $f(c)$ needs to be defined
(2) c need not to be a limit point of A
(3) δ may depend on $\epsilon \& c$
(4) type of definition= ε, δ definition.

Example:

1:
If $c \in A$ is isolated then $f : A \to \mathbb{R}$ is continuous at c
Let $\varepsilon > 0$ Take $\delta > 0$ s.t. $V_\delta(c) \cap A = \{c\}$, then:
$|x - c| < \delta$ and $x \in A \Rightarrow x \in V_\delta(c) \cap A$
$\Rightarrow x = c \Rightarrow f(x) = f(c) \Rightarrow |f(x) - f(c)| = 0 \leq \varepsilon$
2:
$f(x) = x^2$ is continuous at every $c \in \mathbb{R}$
For $|x - c| < 1$ we have $|x| < |c| + 1$ and
$|f(x) - f(c)| = \left|x^2 - c^2\right| = |x + c||x - c| \leq (|x| + |c|)|x - c| < (2|c| + 1)|x - c|$
For a given $\varepsilon > 0$ take $\delta = \min\{1, \frac{\varepsilon}{2|c|+1}\}$
3:
$f(x) = |x|$ is continuous at every $c \in \mathbb{R}$
For al $x, c \in \mathbb{R}$ we have:
$|f(x) - f(c)| = ||x| - |c|| \leq |x - c|$
For a given $\varepsilon > 0$ take $\delta = \varepsilon$
δ independent of c here because constant slope (-1 or 1).

sequential characterization:

$f : A \to \mathbb{R}$ and $c \in A$ Then following statements equivalent.
(1) f continuous @c
(2) $(x_n) \subseteq A$ and $\lim x_n = c \Rightarrow \lim f(x_n) = f(c)$
(3) c limit point of A then 1& 2 also equivalent with $\lim_{x\to c} f(x) = f(c)$

$f : A \to \mathbb{R}$ and $c \in A$ limit point. f not continuous @$x = c$ if there exists $(x_n) \subseteq A$ s.t.
$x_n \neq c \quad \lim x_n = c \quad \lim f(x_n) \neq f(c)$

Example:

there exists no number $a \in \mathbb{R}$ that makes:
$f(x) = \begin{cases} \sin\frac{1}{x} \text{ if } x \neq 0 \\ a \text{ if } x = 0 \end{cases}$ continuous at $x = 0$
(-) if $a \neq 0$, then with $x_n = \frac{1}{n\pi}$ we have: $\lim x_n = 0$ but $\lim f(x_n) = 0 \neq a = f(0)$
(-) if $a = 0$ then with $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ we have $\lim x_n = 0$ but $\lim f(x_n) = 1 \neq a = f(0)$

Dirichlet's function:

Dirichlet's function	Modified dirichlet's function.
$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ Nowhere continuous	$h(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ Only continuous at $x = 0$
PROOF Take $x_n = c + \frac{\sqrt{2}}{n}$ so $x_n \notin \mathbb{Q}$ Then $\lim x_n = c$ but $\lim g(x_n) = 0 \neq g(c)$ Proof of discontinuity at $c \in \mathbb{R} \setminus \mathbb{Q}$ Take $x_n \in \mathbb{Q}$ s.t. $\lvert x_n - c\rvert < \frac{1}{n}, \forall n \in \mathbb{N}$ Then $\lim x_n = c$ But $\lim g(x_n) = 1 \neq g(c)$	PROOF Continuity follows from $\lvert h(x)\rvert \leq \lvert x\rvert$ by: 1.$\lim x_n = 0 \Rightarrow \lim h(x_n) = 0$ or ε, δ definition Proof of discountinuity at $c \neq a$ as for dirichlet's function.

Thomae's function:

$$t(x) = \begin{cases} 1 \text{ if } x = 0 \\ \frac{1}{n} \text{ if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{in lowest terms with } n > 0 \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Discontinuous at each $c \in \mathbb{Q}$ but continuous at each $c \in \mathbb{R} \setminus \mathbb{Q}$
PROOF:
Discontinuity at $c \in \mathbb{Q}$
Take $x_n = c + \frac{\sqrt{2}}{n}$ Then $\lim x_n = c$ but $\lim t(x_n) = 0 \neq t(c)$
Proof of continuity at $c \in \mathbb{R} \setminus \mathbb{Q}$
Let $\varepsilon > 0$ and pick $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$
$(c-1, c+1)$ contains finitely many $r \in \mathbb{Q}$ with denominator $\leq k$
Pick $0 < \delta < 1$ such that $(c - \delta, c + \delta)$ contains no rationals with denominator $\leq k$ then:
$|x - c| < \delta \Rightarrow |t(x) - t(c)| = |t(x)| = t(x) < \frac{1}{k} < \varepsilon$

Lecture 10

Theorem: $f : A \to \mathbb{R}$ continuous and $K \subseteq A$ compact $\Rightarrow f(K)$ compact.
PROOF:
Let $(y_n) \subseteq f(K)$ arbitraru.
$\exists (x_n) \subseteq K$ s.t. $y_n = f(x_n)$ for all n
K compact $\Rightarrow$ some subsequence $x_{n_k} \to x \in K$
f continuous $\Rightarrow y_{n_k} = f(x_{n_k}) \to f(x) \in F(K)$
WARNING: false for pre-images: $f^{-1}(K) = \{x \in A : f(x) \in K\}$
Counter example: $f(x) = 0$ forall $x \in \mathbb{R}$, so K any compact set containing 0, so $f^{-1}(K) = \mathbb{R}$ is not compact.
Theorem maxima and minima:
Let $K \subset \mathbb{R}$ be compact and $f : K \to \mathbb{R}$ continuous then f attains a maximum and a minimum on K
PROOF:

Maximum	Minimum
Exercise 3.3.1 $\Rightarrow s = \sup f(K)$ exists and $s \in f(K)$	Exercise 3.3.1 $\Rightarrow i = \inf f(K)$ exists and $i \in f(K)$
$s = f(c)$ for some $c \in K$	$i = f(c)$ for some $c \in K$
s is an upper bound for $f(K) \Rightarrow f(x) \leq s$ forall $x \in K$	i is a lower bound for $f(K) \Rightarrow f(x) \geq i$ for all $x \in K$

Warning: without compactness previous theorem is false.
Counterexample: $f(x) = x$ no minimum on $(0, 1]$ no maximum on $[0, 1)$ neither a maximum nor a minimum on $\mathbb{R}$
UNIFORM CONTINUOUS $f : A \to \mathbb{R}$ on A if $\forall \varepsilon > 0$,$\exists \delta > 0$ s.t. ,$\forall x, y \in A$:
$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Uniform means that δ does not depend on x or y (but δ may still depend on ε)
NOT UNIFORM CONTINUOUS: $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$,$\exists x, y \in A$ for which ,$|x - y| < \delta$,but $|f(x) - f(y)| \geq \varepsilon_0$
Theorem: $f : K \to \mathbb{R}$ continuous and K is compact, then f uniformly continous on K
PROOF:
Let $\varepsilon > 0$ be arbitrary.
For all $c \in K$ there exists $\delta_c > 0$ such that $|x - c| < 2\delta_c \Rightarrow |f(x) - f(c)| < \frac{1}{2}\varepsilon$
$O_c = (c - \delta_c, c + \delta_c)$ with $c \in K$, form an open cover for K
$K \subset O_{c_1} \cup \ldots \cup O_{c_n}$ for some $c_1, \ldots, c_n \in K$
Take $x, y \in K$ with $|x - y| < \delta = \min\{\delta_{c_1}, \ldots, \delta_{c_n}\}$
$|x - c_i| < \delta_{c_i}$ for some $i = 1, \ldots, n$
$|f(x) - f(c_i)| < \frac{1}{2}\varepsilon$
$|c_i - y| \leq |c_i - x| + |x - y| < \delta_{c_i} + \delta < 2\delta_{c_i}$
$|f(c_i) - f(y)| < \frac{1}{2}\varepsilon$
Apply triangle inequality with $|f(x) - f(c_i)| < \frac{1}{2}\varepsilon$ and $|f(c_i) - f(y)| < \frac{1}{2}\varepsilon$
$\Rightarrow |f(x) - f(y)| < \varepsilon$

Examples:

1:
$f(x) = ax + b$ is uniformly continuous on $\mathbb{R}$
For $x, y \in \mathbb{R}$ we have:
$|f(x) - f(y)| = |(x + b) - (ay + b)| = |a||x - y|$
Let $\varepsilon > 0$ and pick $\delta = \frac{\varepsilon}{|a|}$ then for all $x, y \in \mathbb{R}$ we have:
$|x - y| < \delta \Rightarrow |f(x) - f(y)| < |a|\delta = \varepsilon$
When $a = 0$ we can choose any δ
2:
$f(x) = x^2$ is uniformly continuous on $[a, b]$
For $x, y \in [a, b]$ we have:
$|f(x) - f(y)| = |x + y||x - y| \leq (|x| + |y|)|x - y| \leq 2M|x - y|$ where $M := \max\{|a|, |b|\}$
For $\varepsilon > 0$ take $\delta = \frac{\varepsilon}{2M}$ then for all $x, y \in [a, b]$ we have:
$|x - y| < \delta \Rightarrow |f(x) - f(y)| < 2M\delta = \varepsilon$
3:
$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$
$x_n = n + \frac{1}{n}$ and $y_n = n$
$|x_n - y_n| = \frac{1}{n} \to 0$
$|f(x_n) = f(y_n)| = 2 + \frac{1}{n^2} > 2$ and $\forall n \in \mathbb{N}$
4:
$f(x) = \frac{1}{x}$ is uniform continuous on $[a, \infty)$ for all $a > 0$
For $x, y \in [a, \infty)$ we have:
$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2}$
For $\varepsilon > 0$ take $\delta = a^2\varepsilon$ then for all $x, y \in [a, \infty)$ we have $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\delta}{a^2} = \varepsilon$
5:
$f(x) = \frac{1}{x}$ is not unif. cont. on $(0, \infty)$
$x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n}$
$|x_n - y_n| \to 0$
$|f(x_n) - f(y_n)| = 1$,$\forall n \in \mathbb{N}$
6:
$\sqrt{x}$ is uniformly continuous on $[1, \infty)$
For $x, y \geq 1$ we have:
$\left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2}$
For given $\varepsilon > 0$ take $\delta = 2\varepsilon$ to satisfy the definition.
7:
$[0, 1]$ is compact and $f(x) = \sqrt{x}$ continuous on $[0, 1]$ gives the conclusion that f is continuous on $[0, 1]$

Intermediate value theorem:

$f : [a, b] \to \mathbb{R}$ continuous and $f(a) < L < f(b)$ or $f(a) > L > f(b)$ then $f(c) = L$ for some $c \in (a, b)$
Note: Without loss of generality we can assume
(-) $L = 0$ otherwise replace $f(x)$ by $f(x) - L$
(-) $f(a) < 0 < f(b)$, otherwise replace $f(x)$ by $-f(x)$
Proof:
$\exists I_n = [a_n, b_n]$ s.t. $f(a_n) < 0 \leq f(b_n)$ so $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ so $\text{Length}(I_n) = \frac{b-a}{2^n}$
So $\exists c \in [a, b]$ so $\exists c \in I_n = [a_n, b_n]$, $\forall n \in \mathbb{N}$
Note that: $|a_n - c| \leq \text{Length}(I_n) \to 0 \mid |b_n - c| \leq \text{Length}(I_n) \to 0$
So $c = \lim a_n = \lim b_n$. Continuity of f implies:
$f(c) = \lim f(a_n) = \lim f(b_n)$
We know $f(a_n) < 0$,and $\forall n \in \mathbb{N}$ so $f(c) \leq 0$
We know $f(b_n) \geq 0$,and $\forall n \in \mathbb{N}$ so $f(c) \geq 0$
Combine $f(c) \leq 0$ and $f(c) \geq 0$ we receive $f(c) = 0$

Example:

1:
$p(x) = x^5 - 2x^3 - 2$ has a zero on $(0, 2)$
p is continuous on $[0, 2]$
$p(0) = -2 < 0$ and $p(2) = 14 > 0$
IVT$\Rightarrow p(c) = 0$ for some $c \in (0, 2)$
2:
if $f : [a, b] \to \mathbb{R}$ is continuous and $f([a, b]) \subset [a, b]$, then $f(c) = c$ for some $c \in [a, b]$
Assume $f(a) \neq a$ and $f(b) \neq b$ (Otherwise nothing to prove)
$f([a, b]) \subset [a, b] \Rightarrow f(a) > a, f(b) < b$
$g(x) = f(x) - x$ is continuous and $g(b) < 0 < g(a)$
IVT $\Rightarrow g(c) = 0$ for some $c \in (a, b)$

Lecture 11

Derivative

DERIVATIVE: limit of a difference quotient, denoted by $f'(x)$
DIFFERENTIABLE $f : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$, interval) @$c \in I$ if $f'(c) := \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.
Theorem: $f : I \to \mathbb{R}$ differentiable at $c \in I \Rightarrow f$ continuous at c
PROOF:
$\lim\limits_{x\to c}[f(x) - f(c)] = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot (x - c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim\limits_{x\to c}[x - c] = f'(c) \cdot 0 = 0$

Example:

1:
$f(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$ is not differentiable at $c = 0$. Reason: f is not continuous at $c = 0$
2:
$f(x) = |x|$ continuous but not differentiable at $c = 0$
$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.
3:
f is differentiable at every $c \neq 0$ and $f'(c) = \begin{cases} 1 \text{ if } c > 0 \\ -1 \text{ if } c < 0 \end{cases}$ where $f(x) = |x|$
4:
$f(x) = \frac{x}{1+|x|} \Rightarrow f'(0) = 1$
We can not use the quotient rule, because derivative of $|x|$ where $x = 0$, does not exist.
$\left|\frac{f(x)-f(0)}{x-0} - 1\right| = \left|\frac{1}{1+|x|} - 1\right| = \left|\frac{|x|}{1+|x|} - 1\right| = \frac{|x|}{1+|x|} \leq |x|$
$f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 1$, by ε, δ-argument.
Remark: for $c \neq 0$ we can compute $f'(c)$ using calculus rules.

Theorems:

Name	Theorem	Proof	
Interior Extremum theorem	Assume: $f(a,b) \to \mathbb{R}$ differentiable f attains a maximum or minimum at $c \in (a,b)$ Then $f'(c) = 0$	Maximum: $f(c) \geq f(x)$ for all $x \in (a,b)$ $(x_n)\&(y_n) \in (a,b)$ s.t. $x_n < c < y_n, \forall n \in \mathbb{N}$ and $\lim x_n = \lim y_n = c$ $f'(c) = \frac{f(x_n)-f(c)}{x_n-c} \geq 0$ $f'(c) = \frac{f(y_n)-f(c)}{y-c} \leq 0$ $f'(c) = 0$ by order limit theorem	May be false for closed intervals: $f(x) = x$ on $[0,1]$ min@$x=0$,but $f'(0) = 1$ max@$x=1$ but $f'(1) = 1$
Darboux's Theorem	If $f : [a,b] \to \mathbb{R}$ differentiable $f'(a) < L < f'(b)$ or $f'(a) > L > f'(b)$ there exists $c \in (a,b)$ s.t. $f'(c) = L$	$f'(a) < 0 < f'(b)$ (or replace $f(x)$ by $\pm(f(x) - Lx)$) $\exists s \in (a,b)$ s.t $f(s) < f(a)$ Otherwise $f(x) \geq f(a) \forall x \in (a,b)$ so $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \geq 0$ contradiction can do the same for $f(t) < f(b)$ $[a,b]$ compact, f continuous f mimimum on $[a,b]$ $f(s) < f(a) \& f(t) < f(b) \Rightarrow$ f minimum in (a,b) IET, $f'(c) = 0$ for some $c \in (a,b)$	do not assume f' continuous
Rolle's theorem	Assume that $f : [a,b] \to \mathbb{R}$ and differentiable on (a,b) $f(a) = f(b)$ $\exists c \in (a,b)$ s.t. $f'(c) = 0$	f continous and $[a,b]$ compact so f attains max/min values. $f(a) = f(b)$ both max and min: f constant $\Rightarrow f'(x) = 0$ for all x take any $c \in (a,b)$ otherwise by IET	
Mean Value Theorem	if $[a,b] \to \mathbb{R}$ continuous and f differentiable on (a,b) $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$	$h(x) = f(x) - k(x)$ $k(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ $h(x)$ con. on $[a,b]$ and diff. on (a,b) $h(a) = h(b) = 0$ $h'(c) = 0 \Rightarrow f'(c) = k'(c)$ $f'(c) = \frac{f(b)-f(a)}{b-a}$	

Example:

$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ is NOT derivative.

Assume there exists $F : \mathbb{R} \to \mathbb{R}$ s.t. $F'(x) = f(x)$

Darboux $\Rightarrow f$ attains all values in $(0,1)$

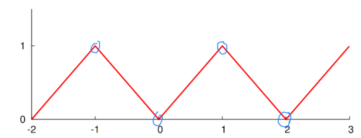
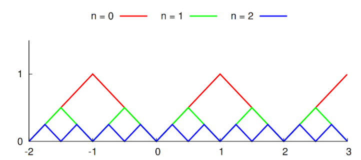
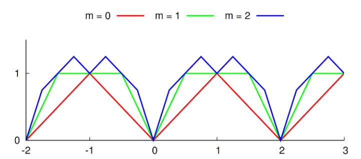
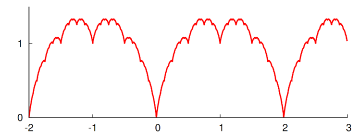
Contradiction!!

Application to uniform continuity

Example:

$f(x) = \arctan(x)$ is uniformly continuous on $\mathbb{R}$
MVT$\Rightarrow \forall x, y \in \mathbb{R}, \exists c \in (x, y)$ s.t.,$\arctan(x) - \arctan(y) = \arctan'(c)(x - y)$
$\arctan(x) - \arctan(y) = \frac{1}{1+c^2}(x - y)$
$|\arctan(x) - \arctan(y)| \leq |x - y|$
For $\varepsilon > 0$ take $\delta = \varepsilon$ to satisfy the definition of uniformly continuity.

Pathologies:

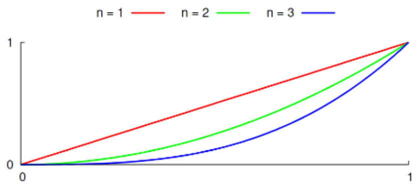
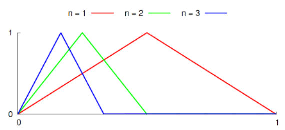
Formula	Graph
$h : \mathbb{R} \to \mathbb{R}$ $h(x) = \lvert x \rvert$ for $x \in [-1, 1]$ $h(x + 2) = h(x)$ for all $x \in \mathbb{R}$	
$h_n(x) = \frac{1}{2^n} h(2^n x)$	
$g_m(x) = \sum_{n=0}^{m} h_n(x)$ $m \to \infty$	
$g(x) = \sum_{n=0}^{\infty} h_n(x)$	

Everywhere continuous, nowhere differentiable.

Lecture 12

SEQUENCE OF FUNCTIONS: $f_n : A \to \mathbb{R}$
f_n POINTWISE CONVERGENCE: to $f : A \to \mathbb{R}$ for all fixed $x \in A$ when $\lim f_n(x) = f(x)$
So for fixed $x \in A$: $\forall \varepsilon > 0 \exists N_{\varepsilon,x} \in \mathbb{N}$ s.t. $n \geq N_{\varepsilon,x} \Rightarrow |f_n(x) - f(x)| < \varepsilon$
f_n UNIFORM CONVERGENCE: to $f : A \to \mathbb{R}$ if:
$\forall \varepsilon > 0$, there $\exists N_\varepsilon \in \mathbb{N}$ s.t. $n \geq N_\varepsilon \Rightarrow |f_n(x) - f(x)| < \varepsilon \forall x \in A$
Note: independent of $x \in A$

Familiar examples:

Name	Picture	proof
Classical Example		$f_n(x) = x^n$ $f(x) = \lim f_n(x) = \begin{cases} 0 \text{ if } x < 1 \\ 1 \text{ if } x = 1 \end{cases}$ on $A = [0,1]$ $\varepsilon > 0$ arbitrary. $x = 0 \vee x = 1$, take $N_{\varepsilon,x} = 1$ $n \geq N_{\varepsilon,x} \Rightarrow |f_n(x) - f(x)| = 0 < \varepsilon$ $0 < x < 1$ Take $N_{\varepsilon,x} > \frac{\log \varepsilon}{\log x}$ $n \geq N_{\varepsilon,x} \Rightarrow |f_n(x) - f(x)| = |x^n - 0| = x^n < \varepsilon$ Observe how N depends on both ε and x!
Triangle sequence		$f_n(x) = \begin{cases} 2nx \text{ if } 0 \leq x \leq \frac{1}{2n} \\ 2 - 2nx \text{ if } \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 \text{ if } \frac{1}{n} \leq x \leq 1 \end{cases}$ Then $f(x) = \lim f_n(x) = 0$ for all $x \in [0,1]$ $0 < x \leq 1$: take $N_{\varepsilon,x} > \frac{1}{x}$ $n \geq N_{\varepsilon,x} \Rightarrow \frac{1}{n} < x \Rightarrow |f_N(x) - f(x)| = |0 - 0| = 0 < \varepsilon$ Observe how N depends on x! $x = 0$: take $N_{\varepsilon,x} = 1$ $n \geq N_{\varepsilon,x} \Rightarrow |f_n(x) - f(x)| = |0 - 0| = 0 < \varepsilon$

The classic example and the triangle inequality does not converge uniform, because we can find a value of ε for which the statement does not hold, but it must hold for all $\varepsilon > 0$ to converge uniform.

A useful characterization:

Theorem: consider $f_n : A \to \mathbb{R}$ then: $f_n \to f$ uniformly $\Leftrightarrow \lim(\sup_{x\in A} |f_n(x) - f(x)|) = 0$

PROOF:

$\Rightarrow$	$\Leftarrow$
for $\varepsilon > 0$ there $\exists N_\varepsilon \in \mathbb{N}$ s.t. $n \geq N_\varepsilon \Rightarrow \lvert f_n(x) - f(x)\rvert < \varepsilon, \forall x \in A$ So $\sup_{x\in A} \lvert f_n(x) - f(x)\rvert \leq \varepsilon$	For $\varepsilon > 0$ there $\exists N_\varepsilon \in \mathbb{N}$ s.t. $n \geq N_\varepsilon \Rightarrow \sup_{x\in A} \lvert f_n(x) - f(x)\rvert < \varepsilon$ $\Rightarrow \lvert f_n(x) - f(x)\rvert < \varepsilon, \forall x \in A$

Example:

1:	**2:**
On $A = [0,1]$ the sequence $f_n(x) = x^n$ Does not converge uniformly to $f(x) = \begin{cases} 0 \text{ if } x < 1 \\ 1 \text{ if } x = 1 \end{cases}$ Reason:for all $n \in \mathbb{N}$ we have $\sup_{x\in[0,1]} \lvert f_n(x) - f(x)\rvert = \sup_{x\in[0,1]} x^n = 1$	The triangle sequence does not converge uniformly to zero since $\sup_{x\in[0,1]} \lvert f_n(x) - f(x)\rvert = \sup_{x\in[0,1]} f_n(x) = 1$
3:	**4:**
$f_n(x) = (1-x)x^n \to 0$ uniformly on $[0,1]$ Calculus method: $f_n(x)$ maximum@$x_n = \frac{n}{n+1}$ $\sup_{x\in[0,1]} \lvert f_n(x) - 0\rvert = f_n(x_n)$ $= \frac{1}{n+1}(\frac{n}{n+1})^n < \frac{1}{n+1} \to 0$	$f_n(x) = \frac{x^2}{1+nx^2} \to 0$ uniformly on $A = \mathbb{R}$

Preservation of continuity:

Assume $f_n : A \to \mathbb{R}$ satisfies:
(1) $f_n \to f$ uniformly on A (2) f_n is continuous at $c \in A$ for all $n \in \mathbb{N}$
Then f is continuous at c
Moral: uniform convergence preserves continuity!
PROOF:
For,$\varepsilon > 0$ there exist: $N \in \mathbb{N}$ s.t. $|f_N(x) - f(x)| < \frac{1}{3}\varepsilon$, for all $x \in A$
$\delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f_N(x) - f_N(c)| < \frac{1}{3}\varepsilon$
If $|x - c| < \delta$ then:
$|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) = f(c)|$
$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$

Example:

The sequence $f_n(x) = x^n$ does NOT uniformly converge to:
$f(x) = \begin{cases} 0 \text{ if } x < 1 \\ 1 \text{ of } x = 1 \end{cases}$ on the set $A = [0,1]$ because each f_n continuous at $x = -1$ but $\lim f$ not.

Lecture 13

CAUCHY CRITERION: Following statements equivalent:
The following statements are equivalent:
(1) f_n converges uniformly on A
(2) for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ s.t. $n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon, \forall x \in A$
PROOF:

$1 \to 2$	$2 \to 1$
For all $\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t.	for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ s.t.:
$n \geq N_\varepsilon \Rightarrow \|f_n(x) - f(x)\| < \frac{\varepsilon}{2} \forall x \in A$	$n, m \geq N_\varepsilon \Rightarrow \|f_n(x) - f_m(x)\| < \varepsilon, \forall x \in A$
$n, m \geq N_\varepsilon \Rightarrow \|f_n(x) - f_m(x)\|$	$\to f(x) := \lim f_n(x)$,exists $\forall x \in A$
$= \|f_n(x) - f(x) + f(x) - f_m(x)\|$	$n, m \geq N_\varepsilon \Rightarrow f_n(x) - \varepsilon < f_m(x) < f_n(x) + \varepsilon, \forall x \in A$
$\leq \|f_n(x) - f(x)\| + \|f(x) - f_m(x)\|$	
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall x \in A$	$n \geq N_\varepsilon \Rightarrow f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon, \forall x \in A$
So (2)	Where $m \to \infty$
	$n \geq N_\varepsilon \Rightarrow \|f_n(x) - f(x)\| < \varepsilon, \forall x \in A$
	So (1)

UNIFORM CONVERGENCE PRESERVE DIFFERENTIABILITY?
Counter example: $f_n(x) = \sqrt{x^2 + \frac{1}{n}} \to |x|$ uniformly on $[-1, 1]$
Every f_n is differentiable at $x = 0$, but the limit is NOT.
Lemma: assume that:
(1) $f_n : [a, b] \to \mathbb{R}$ differentiable for all n
(2) f'_n converges uniformly on $[a, b]$ (note the prime)
(3) $f_n(x_0)$ converges for some $x_0 \in [a, b]$
Then f_n converges uniformly on $[a, b]$
PROOF:
for each $\varepsilon > 0$ there exists $N_1, N_2 \in \mathbb{N}$ s.t.:
$n, m \geq N_1 \Rightarrow |f'_n(x) - f'_m(x)| < \frac{\varepsilon}{2(b-a)}, \forall x \in [a, b]$ and $n, m \geq N_2 \Rightarrow |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$
Claim: $n, m \geq \max\{N_1, N_2\} \Rightarrow |f_n(x) - f_m(x)| < \varepsilon, \forall x \in [a, b]$
PROOF OF CLAIM:
Apply MVT to $g = f_n - f_m$
$g(x) = g(x) - g(x_0) + g(x_0)$
$g(x) = g'(c)(x - x_0) + g(x_0)$ c between x and x_0
Triangle inequality:
$|g(x)| \leq |g'(c)| \cdot |x - x_0| + |g(x_0)| = |g'(c)| \cdot (b - a) + |g(x_0)|$
$|f_n(x) - f_m(x)| \leq |f'_n(c) - f'_m(c)| \cdot (b - a) + |f_n(x_0) - f_m(x_0)|$
$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2(b-a)} \cdot (b - a) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Theorem: If:
1. $f_n : [a,b] \to \mathbb{R}$ differentiable for all n
2. $f_n' \to g$ uniformly on $[a,b]$
3. $f_n(x_0)$ converges for some $x_0 \in [a,b]$
Then there exists a differentiable $f : [a,b] \to \mathbb{R}$ s.t. $f_n \to f$ uniformly and $f' = g$
Moral: $(\lim f_n)' = \lim(f_n')$

PROOF:

Theorem at the top of this page:

Lemma gives $f_n \to f$ uniformly on $[a,b]$ for some f	Let $c \in [a,b]$ and $\varepsilon > 0$ be arbitary To prove: there exists $\delta > 0$ s.t. $0 < \lvert x-c \rvert < \delta \Rightarrow \left\lvert \frac{f(x)-f(c)}{x-c} - g(c) \right\rvert < \varepsilon$
Part 1a	Part 1b

Proof part 1b:

By using the triangle inequality we find the following 3 parts:
$\exists N \in \mathbb{N}$ and $\delta > 0$ s.t.:

Part	statement	Proof
2a	$\left\lvert \frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} \right\rvert < \frac{\varepsilon}{3}$	$\left\lvert \frac{(f_m(x)-f_n(x))-(f_m(c)-f_n(c))}{x-c} \right\rvert = \lvert f_m'(\alpha) - f_n'(\alpha) \rvert$ $\exists N_1 \in \mathbb{N}$ s.t. $n, m \geq N_1 \Rightarrow \lvert f_m'(x) - f_n'(x) \rvert < \frac{\varepsilon}{3} \, \forall x \in [a,b]$ Order limit theorem with $m \to \infty$ $n \geq N_1 \Rightarrow \left\lvert \frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} \right\rvert \leq \frac{\varepsilon}{3}$
2b	$\lvert f_n'(c) - g(c) \rvert < \frac{\varepsilon}{3}$	$n \geq N_2 \Rightarrow \lvert f_n'(c) - g(c) \rvert < \frac{\varepsilon}{3}$
2c	$\left\lvert \frac{f_n(x)-f_n(c)}{x-c} - f_n'(c) \right\rvert < \frac{\varepsilon}{3}$ for $0 < \lvert x-c \rvert < \delta$	fix $n = \max\{N_1, N_2\}$ and $\delta > 0$ s.t. $0 < \lvert x-c \rvert < \delta \Rightarrow \left\lvert \frac{f_n(x)-f_n(c)}{x-c} - f_n'(c) \right\rvert < \frac{\varepsilon}{3}$

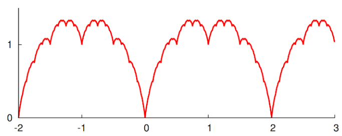
Because we proved statement 2a,2b and 2c, we can say that statement 1b is true, we know that 1a is true (because a direct conclusion from a lemma), and therefore the theorem is true.
SERIES OF FUNCTIONS: Let $f_n : A \to \mathbb{R}$ and $s_n = f_1 + \ldots + f_n$ then:
(-) $\sum_{n=1}^{\infty} f_n \to f$ poinstwise means $s_n \to f$ pointwise.
(-) $\sum_{n=1}^{\infty} f_n \to f$ uniformly means $s_n \to f$ uniformly.

CAUCHY CRITERION: the following statements are equivalent:
(1) $\sum_{n=1}^{\infty} f_n$ converges uniformly on A
(2) for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t.
$n > m \geq N \Rightarrow \lvert f_{m+1}(x) + \ldots + f_n(x) \rvert < \varepsilon$ for all $x \in A$
PROOF:
Follows from: $\lvert s_m(x) - s_n(x) \rvert = \lvert f_{m+1}(x) + \ldots + f_n(x) \rvert$

Weierstrass test: assume that:
(1) $|f_n(x)| \leq C_n$ for all $x \in A$
(2) $\sum_{n=1}^{\infty} C_n$ converges.
Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A
Proof: for all $x \in A$ we have:
$|s_n(x) - s_m(x)| = |f_{m+1}(x) + \ldots + f_n(x)| \leq C_{m+1} + \ldots + C_n$
Cauchy criterion for $\sum_{n=1}^{\infty} C_n \Rightarrow$ Cauchy criterion for s_n
Preservation of continuity: assume:
(1) $\sum_{n=1}^{\infty} f_n \to f$ uniformly on A
(2) f_N is continuous on A for all n
Then f is continuous on A
Proof:
$s_n = f_1 + \ldots + f_n$ is continuous on A for all $n \in \mathbb{N}$
$s_n \to f$ uniformly $\to f$ is continuous on A
Preservation of differentiability: Assume:
(1) $f_n : [a,b] \to \mathbb{R}$ is differentiable for all n
(2) $\sum_{n=1}^{\infty} f_n' \to g$ uniformly on $[a,b]$
(3) $\sum_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in [a,b]$
Then there exists a differentiable $f : [a,b] \to \mathbb{R}$ s.t. $\sum_{n=1}^{\infty} f_n \to f$ uniformly and $f' = \sum_{n=1}^{\infty} f_n'$

Example:

1:



Same graphs as before:
Claim: $f_n(x) ;= \frac{1}{2^n} h(2^n x) \Rightarrow |f_n(x)| \leq \frac{1}{2^n}$ for all $x \in \mathbb{R}$
$\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges.

Weierstrass test $\Rightarrow \sum_{n=0}^{\infty} f_n$ converges uniformly on $\mathbb{R}$
f_n continuous on $\mathbb{R}$ for all $n \in \mathbb{N} \Rightarrow f$ continuous on $\mathbb{R}$
2: $f(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{3^n}$ is differentiable on every $[-c,c]$
(-) $f_n(x) = \sin(2^n x)/3^n$ is differentiable for $n \in \mathbb{N}$
(-) $|f_n'(x)| \leq (\frac{2}{3})^n, \forall x \in [-c,c]$
Weierstrass $\Rightarrow \sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[-c,c]$
(-) $\sum_{n=1}^{\infty} f_n(x)$ converges at $x = 0$ Apply term-wise differentiability Theorem.

Lecture 14

Power series general form: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

Pointwise convergence thm: $\sum_{n=0}^{\infty} a_n x^n$ converges at $c \neq 0 \Rightarrow \sum_{n=0}^{\infty} |a_n x^n|$ converges for $|x| < |c|$

Proof:

$\sum_{n=0}^{\infty} a_n c^n$ converges $\Rightarrow \lim a_n c^n = 0$

$\Rightarrow (a_n c^n)$ is bounded.

$\Rightarrow \exists M > 0$ s.t. $|a_n c^n| \leq M, \forall n \in \mathbb{N}$

$|a_n x^n| = \left|a_n (c \cdot \frac{x}{c})^n\right| = |a_n c^n| \cdot \left|\frac{x}{c}\right|^n \leq M \cdot \left|\frac{x}{c}\right|^n, \forall n \in \mathbb{N}$

Note $|x| < |c| \Rightarrow \left|\frac{x}{c}\right| < 1$

Therefore we see that $|a_n x^n| \leq M$

So Apply comparison test:

$\sum_{n=0}^{\infty} M\left|\frac{x}{c}\right|^n$ converges $\Rightarrow \sum_{n=0}^{\infty} |a_n x^n|$ converges.

Radius of convergence: R when $R \geq 0$

(-) $|x| < R \Rightarrow$ PS converges at x

(-) $|x| > R \Rightarrow$ PS diverges at x **Computing the radius.**

(-) Root test: $L = \lim \sqrt[n]{|a_n|}$ exists then $R = \frac{1}{L}$

(-) Ratio test: $L = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $R = \frac{1}{L}$

(-) $L = 0$? then $R = \infty$

Proof:

$\lim \sqrt[n]{|a_n x^n|} = L|x|, \forall x \in \mathbb{R}$ fixed.

For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \left|\sqrt[n]{|a_n x^n|} - L|x|\right| < \varepsilon$

$\Rightarrow L|x| - \varepsilon < \sqrt[n]{|a_n x^n|} < L|x| + \varepsilon$

$\Rightarrow (L|x| - \varepsilon)^n < |a_n x^n| < (L|x| + \varepsilon)^n$

$x < \frac{1}{L}$	$x > \frac{1}{L}$
Pick $\varepsilon < 1 - L\|x\|$	pick $\varepsilon < L\|x\| - 1$
$L\|x\| + \varepsilon < 1 \Rightarrow \sum_{n=0}^{\infty} (L\|x\| + \varepsilon)^n$ converges.	$L\|x\| - \varepsilon > 1 \Rightarrow (L\|x\| - \varepsilon)^n$ unbounded.
$\Rightarrow \sum_{n=0}^{\infty} \|a_n x^n\|$ converges.	$\Rightarrow \|a_n x^n\|$ unbounded.
$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ converges.	$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ diverges

Example:

Root test:	**Ratio test:**
$\sum_{n=0}^{\infty} \frac{x^n}{5^{n^2}}$ Radius of convergence:	$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$
$a_n = \frac{1}{5^{n^2}} \Rightarrow \sqrt[n]{\|a_n\|} = \frac{1}{5^n}$	$a_n = \frac{1}{n^2} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}$
$\Rightarrow L = 0$	$L = 1$
$\Rightarrow R = \infty$	$R = 1$
$\Rightarrow x < R$	converges for value in closed interval $[-1, 1]$
$\Rightarrow$ PS converges.	

Theorems:

BEWARE OF THE BOUNDARY POINTS:

Example	Radius	at $x=-R$	at $x=R$
$\sum_{n=1}^{\infty} x^n$	$R=1$	divergent	divergent.
$\sum_{n=1}^{\infty} \frac{1}{n}x^n$	$R=1$	convergent	divergent
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}x^n$	$R=1$	divergent	convergent.
$\sum_{n=1}^{\infty} \frac{1}{n^2}x^n$	$R=1$	convergent	convergent

Theorem uniform convergence: $\sum_{n=0}^{\infty} |a_n c^n|$ convergent $\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ uniformly convergent on $[-|c|, |c|]$

PROOF:
For $|x| \leq |c|$ we have: $|a_n x^n| = |a_n| \cdot |x|^n \leq |a_n| \cdot |c|^n = |a_n c^n| =: M_n$
Apply Weierstrass' test: $\sum_{n=0}^{\infty} M_n$ convergent $\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ uniformly convergent on $[-|c|, |c|]$

Continuity of the limit: Corollary: $\sum_{n=0}^{\infty} a_n x^n$ continuous function on $(-R, R)$ PROOF: Take $x_0 \in (-R, R)$ and $|x_0| < c < d < R$ then:
PS convergent at d $\Rightarrow$PS absolutely convergent at c
$\Rightarrow$ PS uniformly convergent on $[-c, c]$ $\Rightarrow$ PS continuous on $[-c, c]$
Each $a_n x^n$ is continuous!
$\Rightarrow$ PS continuous at x_0 $\Rightarrow$ PS continuous on $(-R, R)$
CONTINUITY OF THE LIMIT (2):
$\sum_{n=0}^{\infty} |a_n R^n|$ convergent $\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ uniformly convergent on $[-R, R]$
In particular, the PS is continuous on $[-R, R]$
What if convergence is conditional at $X = R$ or $x = -R$

Lemma: if $s_n = u_1 + \ldots + u_n$ then: $\sum_{k=1}^{n} u_k v_k = s_n v_{n+1} + \sum_{k=1}^{n} s_k(v_k - v_{k+1})$

PROOF:
Set $s_0 = 0$ then: $u_k v_k = (s_k - s_{k-1})v_k = s_k(v_k - v_{k+1}) + s_k v_{k+1} - s_{k-1} v_k$,$\forall k = 1, \ldots, n$
These last two terms are called the telescoping terms. $\sum_{k=1}^{n} u_k v_k = s_n v_{n+1} + \sum_{k=1}^{n} s_k(v_k - v_{k+1})$ **Abel's Lemma:** Assume that (u_n) and (v_n) satisfy:
(1) $|u_1 + \ldots + y_n| \leq C$,$\forall n \in \mathbb{N}$ (2) $0 \leq v_{n+1} \leq v_n$,$\forall n \in \mathbb{N}$
Then $\left|\sum_{k=1}^{n} u_k v_k\right| \leq C v_1$,$\forall n \in \mathbb{N}$

PROOF:
$s_n = u_1 + \ldots + u_1$ so $\left|\sum_{k=1}^{n} u_k v_k\right| = \left|s_n v_{n+1} + \sum_{k=1}^{n} s_k(v_k - v_{k+1})\right|$
$\left|\sum_{k=1}^{n} u_k v_k\right| \leq |s_n| v_{n+1} + \sum_{k=1}^{n} |s_k|(v_k - v_{k+1})$
$\left|\sum_{k=1}^{n} u_k v_k\right| \leq C(v_{n+1} + \sum_{k=1}^{n} (v_k - v_{k+1})) = C v_1$

Abel's theorem:
(1) PS converges at $x = R \Rightarrow$ PS converges uniformly on $[0, R]$
(2) PS converges at $x = -R \Rightarrow$ PS converges uniformly on $[-R, 0]$
PROOF: only part 1:
for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $n > m \geq N \Rightarrow \left|\sum_{k=m+1}^{n} a_k R^k\right| < \varepsilon$
take any $x \in [0, R]$ and set: $v_k = (\frac{x}{R})^k$, then: $u_k = \begin{cases} a_k R^k \text{ if} k \geq m+1 \\ 0 \text{ otherwise} \end{cases}$
Abel's lemma $\rightarrow$ Cauchy criterion: $\left|\sum_{k=m+1}^{n} a_k x^k\right| = \left|\sum_{k=1}^{n} y_k v_k\right| < \varepsilon \cdot \frac{x}{R} \leq \varepsilon \, \forall x \in [0, R]$

Differentiation theorem:
$\sum\limits_{n=0}^{\infty} a_n x^n$ convergent on $(-R,R) \Rightarrow \sum\limits_{n=0}^{\infty} n a_n x^{n-1}$ convergent on $(-R,R)$
Proof:
$|c|<1$ then there exists $M>0$ s.t. $\left|nc^{-1}\right| \leq M, \forall n \in \mathbb{N}$
Let $|x|<t<R$, then: $\left|n a_n x^{n-1}\right| = \frac{1}{t}(n\left|\frac{x}{t}\right|^{n-1})|a_n t^n| \leq \frac{M}{t}|a_n t^n|$
Apply comparison test.
Differentiation term by term:
For any PS with radius R we have: $(\sum\limits_{n=0}^{\infty} a_n x^n)' = \sum\limits_{n=0}^{\infty} n a_n x^{n-1}, \forall x \in (-R,R)$
Proof:
let $0 \leq c < R$ then:
$\sum\limits_{n=0}^{\infty} n a_n x^{n-1}$ converges uniformly on $[-c,c]$ so $\sum\limits_{n=0}^{\infty} a_n x^n$ converges at $x=0$
Now apply Term-wise differentiability Theorem.

Examples:

1:
for all $x \in (-1,1)$ we have:
$\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$
$\sum\limits_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$
Taking $x=\frac{1}{4}$ gives:
$\sum\limits_{n=1}^{\infty} \frac{n}{4^n} = \frac{1}{4}\sum\limits_{n=0}^{\infty} n(\frac{1}{4})^{n-1} = \frac{1}{4}\cdot\frac{1}{(1-\frac{1}{4})^2} = \frac{4}{9}$
2:
For all $x \in (-1,1)$ we have:
$\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \to f(x)$
$\sum\limits_{n=1}^{\infty} (-1)^{n+1} x^{n-1} \to f'(x) = \frac{1}{1+x} \Rightarrow f(x) = \log|1+x| + C$
Note that
(-) $C = f(0) = 0$ so $f(x) = \log|1+x|$
(-) Abel's Theorem: $\Rightarrow$ PS in the original equation uniformly on $[0,1]$
(=) Hence, PS in original equation is continupus at $x=1$
$\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \lim\limits_{x\to1} \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \lim\limits_{x\to1} f(x) = f(1) = \log(2)$
Conclusion: $\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Lecture 15

TAYLOR SERIES of f around $x = 0$: given by: $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

PARTIAL SUM: $s_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^l$

REMAINDER: $E_n(x) = f(x) - s_n(x)$

Lemma: t variable, x fixed. Assume that:

(-) $x > 0$ and $h(t)$ is $n+1$ times differentiable on $[0, x]$

(-) $h(x) = 0$ and $h^{(k)}(0) = 0$ for all $k = 0, \dots, n$

Then $h^{(n+1)}(c) = 0$ for some $c \in (0, x)$

PROOF:

Repeated application Rolle's theorem:

$h(0) = h(x) \Rightarrow h'(c_1) = 0$ for some $c_1 \in (0, x)$

$h'(0) = h'(c_1) \Rightarrow h''(c_2) = 0$ for some $c_2 \in (0, c_1)$

$\vdots$

$h^{(n)}(0) = h^{(n)}(c_n) \Rightarrow h^{(n+1)}(c_{n+1}) = 0$ for some $c_{n+1} \in (0, c_n)$

Theorem:

for $n \in \mathbb{N}$ and $x > 0$, there exists $c \in (0, x)$ s.t.: $E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

Note: c depends on both n and x!

PROOF:

Fix $x > 0$ and consider: $h(t) = f(t) - s_n(t) - (\frac{f(x) - s_n(x)}{x^{n+1}}) t^{n+1}$

note that $h(x) = 0$ and $h^{(k)}(0) = 0$ for $k = 0, \dots, n$

Previous lemma gives $c \in (0, x)$ s.t.: $f^{(n+1)}(c) - s_n^{(n+1)}(c) - (n+1)!(\frac{f(x) - s_n(x)}{x^{n+1}}) = 0$

We can claim that $s_n^{(n+1)}(c) = 0$

$f(x) - s_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

TAYLOR SERIESof f around $x = a$: $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$

LAGRANGE REMAINDER: for $x > a$ exists $c \in (a, x)$ s.t. $E_n(x) = f(x) - s_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$

Examples:

Euler:
Taylor series for $f(x) = e^x$

n	$f^{(n)}(x)$	$a_N = f^{(n)}(0)/n!$
0	e^x	e^x
1	e^x	e^x
2	e^x	$e^x/2!$
$\vdots$	$\vdots$	$\vdots$

For $x \neq 0$ there exists $c \in (-|x|, |x|)$ s.t.:

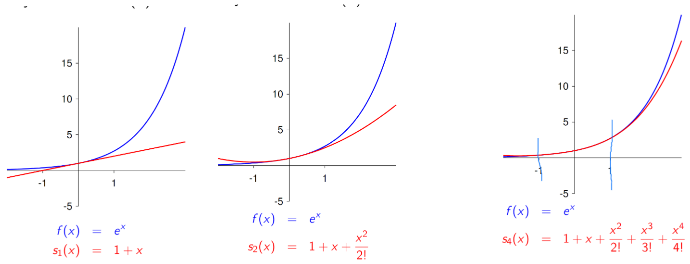
$$e^x = \sum_{k=0}^{n} \frac{1}{k!} x^k + \frac{e^c}{(n+1)!} x^{n+1}$$

For any $a > 0$ we have:

$$\sup_{x \in [-a,a]} \left| e^x - \sum_{k=0}^{n} \frac{1}{k!} x^k \right| \leq e^a \cdot \frac{a^{n+1}}{(n+1)!} \to a \text{ as } n \to \infty$$

The taylor series of f converges to f on $[-a, a]$!

Graph:



. . .

sin(x)

n	$f^{(n)}(x)$	$a_n = f^{(n)}(0)/n!$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	$-\frac{1}{3!}$
4	$\sin(x)$	0
5	$\cos(x)$	$\frac{1}{5!}$
$\vdots$	$\vdots$	$\vdots$

For $x \neq 0$ there exists $c \in (-|x|, |x|)$ s.t.

$$|E_n(x)| = \left| \frac{f^{n+1}(c)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Remainder converges to 0 uniformly on any interval $[-a, a]$:

$$\sup_{x \in [-a,a]} |E_n(x)| \leq \frac{a^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

Conclusion:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots, \forall x \in \mathbb{R}$$

When we make a graph of these taylor series, we see that the taylor series approxiomate the sinfunction better for every higher value of n

Natural logarithm:
$f(x) = \ln(1 + x) \Rightarrow f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}\ \forall n \in \mathbb{N}$

For $x > 0$ exists $c \in (0, x)$ s.t.: $\ln(1 + x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k + \frac{(-1)^n}{(n+1)(1+c)^{n+1}} x^{n+1}$

arctan(x)
On $[-1, 1]$ we have $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$
The convergence is uniform on $[0, 1]$ but not on $[-1, 0]$
For $x = 1$ we get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Counterexample:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} \Rightarrow f^n(0) = 0\ \forall n \in \mathbb{N}$$

The Taylor series of f does not converge to f

Applications:

$\int_0^1 \frac{e^x-1}{x} dx \approx 1.3179$ accoarding Wolfram Alpha.
Approximating square roots by an example:
$\sqrt{x}$ centered at $x = 1$ $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + E_3(x)$
This gives $\sqrt{5} \approx 1$ which is not true.
Centered at $x = 2$ $\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + E_3(x)$
Then $\sqrt{5}$ gives 2.234375, which is really close to the real value.

Approximating integrals:

For $x > 0$ exists $c \in (0, x)$ s.t.:

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c}{(n+1)!} x^{n+1}$$

$$\frac{e^x-1}{x} = \sum_{k=1}^{n} \frac{x^{k-1}}{k!} + \frac{e^c}{(n+1)!} x^n$$

$$\int_0^1 \frac{e^x-1}{x} dx = \sum_{k=1}^{n} \frac{1}{k!k} + \int_0^1 \frac{e^c}{(n+1)!} x^n dx$$

Upper bound Right part: $R_n = \int_0^1 \frac{e^c}{(n+1)!} x^n dx$

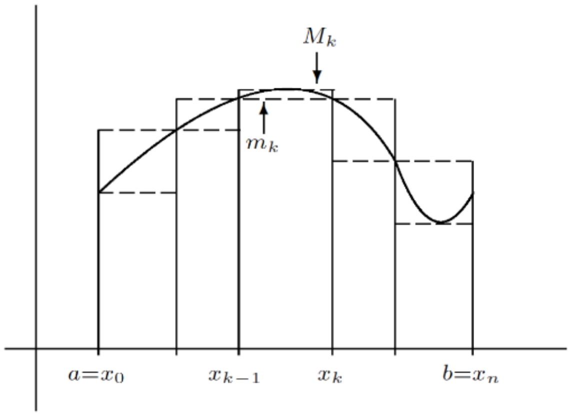
$$\int_0^1 \frac{e^c}{(n+1)!} x^n dx < \int_0^1 \frac{3}{(n+1)!} x^n dx = \frac{3}{(n+1)!(n+1)}$$

When we fill it in again we see that: $\int_0^1 \frac{e^x-1}{x} dx \approx \sum_{k=1}^{5} \frac{1}{k!k} = 1.31763\ldots$ Where $(R_5 < 0.001)$

Lecture 16

PARTITION: a partition of $[a, b]$ is a set of the form: $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$
REFINEMENTS: Q refinement of P if $P \subseteq Q$ provided that P and Q partitions same interval.

Let $f : [a, b] \to \mathbb{R}$ be bounded and P be a partition of $[a, b]$ then:
LOWER SUM of f w.r.t P: $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
Approximate area below graph of f $L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$
UPPER SUM of f w.r.t P: $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$
Approximate area above graph of f $U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$



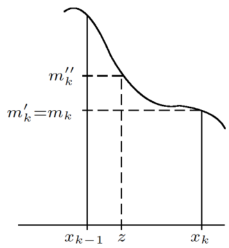
$L(f, P) \leq U(f, P)$ for any partition P of $[a, b]$

Example:

1:
$P_1 = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ partition of $[0, 1]$
$P_2 = \{0, 1, 2\}$ NOT partition of $[0, 1]$
$P_3 = \{0, \frac{1}{2}\}$ NOT partition of $[0, 1]$
2:
$P = \{0, \frac{1}{2}, 1\}$ partition $[0, 1]$
$Q_1 = \{-, \frac{1}{4}, \frac{1}{2}, \frac{9}{10}, 1\}$ refines P
$Q_2 = \{0, \frac{1}{2}, 1, 2\}$ does not refine P because $2 \notin [0, 1]$

Relation upper and lower sums:

Lemma: if $P \subseteq Q$ then:
(-) $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$
(-) $U(f,Q) - L(f,Q) \leq U(f,P) - L(f,P)$ PROOF:



Only proof upper sum, lower soom works the samae way.

Refine P by adding one point $z \in [x_{k-1}, x_k]$
$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
$m_k' = \inf\{f(x) : x \in [z, x_k]\}$
$m_k'' = \inf\{f(x) : x \in [x_{k-1}, z]\}$
We know that $A \subset B$ then $\inf A \geq \inf B$
$m_k(x_k - x_{k-1}) = m_k(x_k - z) + m_k(z - xk - 1) \leq m_k'(x_k - z) + m_k''(z - x_{k-1})$
Then proceed by induction
Lemma: for any two partitions P_1 and P_2 we have: $L(f,P_1) \leq U(f,P_2)$
PROOF: $Q = P_1 \cup P_2$ then $P_1, P_2 \subset Q$, so: $L(f,P_1) \leq L(f,Q) \leq U(f,Q) \leq U(f,P_2)$

Best possible approximate area and riemann integral:

Assume $f : [a,b] \to \mathbb{R}$ is bounded.
Let $\mathcal{P}$ denote the collections of all partitions fo $[a,b]$
$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}$ $L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}$
Lemma: $L(f) \leq U(f)$
PROOF:
$L(f,P_1) \leq U(f,P_2)$ for all $P_1, P_2 \in \mathcal{P}$
$L(f) \leq U(f,P_2)$ for all $P_2 \in \mathcal{P}$(take sup over P_1)
$L(f) \leq U(f)$ (Take inf over P_2)

RIEMANN INTEGRABLE: bounded function $f : [a,b] \to \mathbb{R}$ and $U(f) = L(f)$
Notation: $\int_a^b f = U(f) = L(f)$ or $\int_a^b f(x)dx = U(f) = L(f)$

Integrability:

Theorem: The following statements are equivalent:
(1) f is integrable.
(2) or all $\varepsilon > 0$ there exists a partition P_ε s.t. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$
PROOF:
(2) $\Rightarrow$ (1)
$\left.\begin{matrix} U(f) \leq U(f, P_\varepsilon) \\ L(f) \geq L(f, P_\varepsilon) \end{matrix}\right\} \Rightarrow U(f) - L(f) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$
This holds for all $\varepsilon > 0$ so $U(f) = L(f)$
(1) $\Rightarrow$ (2)
let $\varepsilon > 0$ and choose P_1 and P_2 such that:
$L(f, P_1) > L(f) - \frac{1}{2}\varepsilon$ and $U(f, P_2) < U(f) + \frac{1}{2}\varepsilon$
Because of the characterizations of infimum and supremium.
Let $P_\varepsilon = P_1 \cup P_2$ then:
$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq U(f, P_2) - L(f, P_1) = [U(f, P_2) - U(f)] + [L(f) - L(f, P_1)] < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$
So $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$
Continuous functions: f continuous on $[a, b] \Rightarrow f$ integrable on $[a, b]$
PROOF:
f is uniformly continuous on $[a, b]$
For all $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$ for all $x, y \in [a, b]$
Let P be a partition such that $x_k - x_{k-1} < \delta$ for all $k = 1, \ldots, n$
There exists $y_k, z_k \in [x_{k-1}, x_k]$ s.t. $f(y_k) = M_k$ and $f(z_k) = m_k$
Note: $|y_k - z_k| < \delta \Rightarrow M_k - m_k = f(y_k) - f(z_k) < \frac{\varepsilon}{b-a}$
$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1})$
$= \frac{\varepsilon}{b-a} \cdot (x_n - x_0) = \frac{\varepsilon}{b-a}(b - a) = \varepsilon$
So $U(f, P) - L(f, P) < \varepsilon$ So integrable.

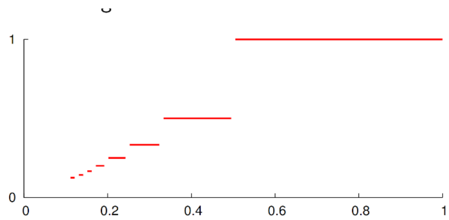
Example:

$f(x) = \begin{cases} 1 \text{ if } x \neq 1 \\ 0 \text{ if } x = 1 \end{cases}$ is integrable on $[0, 2]$
Let $0 < \varepsilon < 1$ and take the partition: $P = \{0, 1 - \frac{1}{3}\varepsilon, 1 + \frac{1}{4}\varepsilon, 2\}$
$U(f, P) = 2$ and $L(f, P) = 2 - \frac{1}{2}\varepsilon$ so $U(f, P) - L(f, P) < \varepsilon$
2:
$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ is not integrable on $[0, 1]$
Let P be any partition of $[0, 1]$ then:
$[x_k, x_{k-1}] \cap \mathbb{Q}^c \neq \emptyset \Rightarrow m_l = 0$ for all $k = 1, \ldots, n \Rightarrow L(f, P) = 0$
$[x_k, x_{k-1}] \cap \mathbb{Q} \neq \emptyset \Rightarrow M_k = 1$ for all $k - 1, \ldots, n \Rightarrow U(f, P) = 1$
So $L(f, P) \neq U(f, P)$ and therefore not differentiable.
3:
$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ is NOT integrable on $[0, 1]$
for any partition P of $[0, 1]$ we have: $U(f, P) - L(f, P) =$
$\sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k(x_k - x_{k-1}) > \sum_{k=1}^{n} \frac{1}{2}(x_k + x_{k-1})(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{1}{2}(x_k^2 - x_{k-1}^2) = \frac{1}{2}$

Increasing functions:

Any increasing function $f : [a, b] \to \mathbb{R}$ integrable.
For any partition of $[a, b]$ we have:
$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_k)$
$m_k = \inf\{f(x) : x \in [x_{k-1}, x_l]\} = f(x_{k-1})$
An equispaced partition P gives:
EQUISPACED: Every interval has the same size.
$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \frac{b-a}{n} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]$
$= \frac{(b-a)(f(b)-f(a))}{n} \to 0 \text{ as } n \to \infty$

Example:



$f(x) = \begin{cases} 0 \text{ if } x = 0 \\ \frac{1}{p} \text{ if } x \in (\frac{1}{p+1}, \frac{1}{p}] \text{ for some } p \in \mathbb{N} \end{cases}$
Since f is increasing it is integrable on $[0, 1]$

Lecture 17:

Split property: $f : [a,b] \to \mathbb{R}$ bounded and $c \in (a,b)$ then f integrable on $[a,b] \Leftrightarrow f$ integrable on $[a,c]$ and $[c,b]$. In that case: $\int\limits_a^b f = \int\limits_a^c f + \int\limits_c^b f$

Proof

Part 1:

Let $\varepsilon > 0$, and pick a paritition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Let $P_c = P \cup \{c\}$ then: $U(f,P_c) - L(f,P_c) < \varepsilon$

P_c is in fact the original partition where we add the point c

Then $Q = P_c \cap [a,c]$ is a partition of $[a,c]$ and:

$$\left.\begin{array}{l} m := \#\text{intervals in } Q \\ n := \#\text{intervals in } P_c \end{array}\right\} \Rightarrow m < n$$

$m < n$ implies:

$$U(f,Q) - L(f,Q) = \sum_{k=1}^{m}(M_k - m_k)(x_k - x_{k-1}) \le \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = U(f,P_c) - L(f,P_c) < \varepsilon$$

So $U(f,P_c) - L(f,P_c) < \varepsilon$, conclusion f integrable on $[a,c]$

Part 2:

Let P_1 and P_2 partititions of $[a,c]$ and $[c,b]$ s.t.: $U(f,P_i) - L(f,P_i) < \frac{1}{2}\varepsilon$ for $i = 1,2$

Then $P = P_1 \cup P_2$ is a partition of $[a,b]$ and:

$U(f,P) = U(f,P_1) + U(f,P_2)$

$L(f,P) = L(f,P_1) + L(f,P_2)$

$U(f,P) - L(f,P) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$

Conclusion: f integrable on $[a,b]$

Part 3: Let ε and P_1, P_2 be as before:

$$\int\limits_a^b f \le U(f,P) < L(f,P) + \varepsilon = L(f,P_1) + L(f,P_2) + \varepsilon \le \int\limits_a^c f + \int\limits_c^b f + \varepsilon$$

So we can claim: $\int\limits_a^b f \le \int\limits_a^c f + \int\limits_c^b f$

Because: $x \le y + \varepsilon$, for $\varepsilon > 0$ then $x \le y$

Part 4:

Let $\varepsilon > 0$ and P_1, P_2 be as before:

$$\int\limits_a^c f + \int\limits_c^b \le U(f,P_1) + U(f,P_2) < L(f,P_1) + f, P_2 + \varepsilon = L(f,P) + \varepsilon \le \int_a^b f + \varepsilon$$

So we have $\int\limits_a^c f + \int\limits_c^b \le \int\limits_a^b f$

And because we have: $\int\limits_a^b f \le \int\limits_a^c f + \int\limits_c^b f$ And: $\int\limits_a^c f + \int\limits_c^b \le \int\limits_a^b f$ we proved it.

Integrable, algebraic properties and order properties:

f INTEGRABLE ON A CLOSED INTERVAL $[a,b]$: $\int\limits_a^b f = -\int\limits_b^a f$ and $\int\limits_c^c f = 0$ for all $c \in [a,b]$

Corollary: regardless order a, b, c we have: $\int\limits_a^b f = \int\limits_a^c f + \int\limits_c^b f$

Algebraic properties: If f, g integrable on $[a,b]$ then:

1. $f+g$ integrable and $\int\limits_a^b (f+g) = \int\limits_a^b f + \int\limits_a^b g$

2. kf integrable and $\int\limits_a^b kf = k\int\limits_a^b f$ for all $k \in \mathbb{R}$

Order properties:

(1) f integrable on $[a,b]$ then $m \leq f(x) \leq M \Rightarrow m(b-a) \leq \int\limits_a^b f \leq M(b-a)$

(2) f, g integrable on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a,b]$ then $\int\limits_a^b f \leq \int\limits_a^b g$

(3) f integrable on $[a,b]$ then $|f|$ integrable and $\left|\int\limits_a^b f\right| \leq \int\limits_a^b |f|$

PROOF:

(1) For all partitions of $[a,b]$, we have $L(f,P) \leq \int\limits_a^b f \leq U(f,P)$

Taking $P = \{a,b\}$ gives:
$U(f,P) = (b-a) \cdot \sup\{f(x) : x \in [a,b]\} \leq M(b-a)$
$L(f,P) = (b-a) \cdot \inf\{f(x) : x \in [a,b]\} \geq m(b-a)$

(2) Since $0 \leq g(x) - f(x)$ for all $x \in [a,b]$ we have: $0 \cdot (b-a) \leq \int\limits_a^b (g-f) \Rightarrow 0 \leq \int\limits_a^b g - \int\limits_a^b f$

(3) P any partition of $[a,b]$ and:
$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
$M'_k = \sup\{|f(x)| : x \in [x_{k-1}, x_k]\}$ $m'_k = \inf\{|f(x)| : x \in [x_{k-1}, x_k]\}$
Claim: $M'_k - m'_k \leq M_k - m_k$
For all $\varepsilon > 0$ exists $y, z \in [x_{k-1}, x_k]$ s.t.
$M'_k - \frac{1}{2}\varepsilon < |f(y)|$
$m'_k + \frac{1}{2}\varepsilon > |f(z)|$
$M'_k - m'_k - \varepsilon < |f(y)| - |f(z)| \leq |f(y) - f(z)| \leq M_k - m_k$ so $M'_k - m'_k \leq M_k - m_k$
$U(|f|,P) - L(|f|,P) = \sum\limits_{k=1}^n (M'_k - m'_k)(x_k - x_{k-1}) \leq \sum\limits_{k=1}^n (M_k - m_k)(x_k - x_{k-1}) = U(f,P) - L(f,P) < \varepsilon$
Hence f integrable $\Rightarrow |f|$ integrable.

$-|f(x)| \leq f(x) \leq |f(x)| \Rightarrow -\int\limits_a^b |f| \leq \int\limits_a^b f \leq \int\limits_a^b |f| \Rightarrow \left|\int\limits_a^b f\right| \leq \int\limits_a^b |f|$

The fundamental theorem

Part 1:
Assume that:
(1) f is integrable on $[a,b]$ (2) F differentiable on $[a,b]$ and $F'(x)=f(x)$,$\forall x\in[a,b]$

Then $\int\limits_a^b f = F(b)-F(a)$

Part 2:
Let f integrable on $[a,b]$ and define:
$F(x)=\int_a^x F(t)dt$ where $x\in[a,b]$
Then:
(1) F uniformly continuous on $[a,b]$
(2) If f is continuous at c then F is differentiable at c and $F'(c)=f(c)$

Proof part 1:
Let P be any partition of $[a,b]$:
$F(b)-F(a)=\sum\limits_{k=1}^{n}[F(x_k)-F(x_{k-1})]$
Because: $F(b)-F(a)=F(x_n)-F(0)$
MVT where $t_k\in(x_{k-1},x_k)$:
$\sum\limits_{k=1}^{n} f(t_k)(x_k-x_{k-1})<\sum\limits_{k=1}^{n} M_k(x_k-x_{k-1})=U(f,P)$
$F(b)-F(A)\geq L(f,P)$ by similar proof, so we have:
$L(f,P)\leq F(b)-F(a)\leq U(f,P)$
Taking sup/inf over all partitions gives:
$L(f)\leq F(b)-F(a)\leq U(f)$
Since f integrable, it follows that:
$L(f)=U(f)=F(b)-F(a)$

Proof Part 2:
Statement 1:
since f integrable on $[a,b]$ there exists $M>0$ s.t.: $|f(x)|\leq M\,\forall x\in[a,b]$
We can not compute integrals of unbounded functions so that is the reason we can say that.
If $x,y\in[a,b]$ with $x\geq y$ then:
$|F(x)-F(y)|=\left|\int\limits_y^x f(t)dt\right|\leq\int\limits_y^x|f(t)|dt\leq M|x-y|$
For given $\varepsilon>0$ take $\delta=\frac{\varepsilon}{M}$ So therefore, F uniformly continuous on $[a,b]$
Statement 2:
for $x\neq c$ we have:
$\frac{F(x)-F(c)}{x-c}-f(c)=\frac{1}{x-c}\int\limits_c^x f(t)dt-f(c)=\frac{1}{x-c}\int\limits_c^x f(t)-f(c)dt$ Let $\varepsilon>0$ be arbitrary and pick $\delta>0$ s.t.:
$|x-c|<\delta\Rightarrow|f(x)-f(c)|<\varepsilon$
Since $|t-c|\leq|x-c|<\delta$ it follows: $\left|\frac{F(x)-F(c)}{x-c}-f(c)\right|=\frac{1}{|x-c|}\left|\int\limits_c^x f(t)-f(c)dt\right|\leq\frac{1}{|x-c|}|x-c|\cdot\varepsilon=\varepsilon$
So $\left|\frac{F(x)-F(c)}{x-c}-f(c)\right|<\varepsilon$